CHAPTER 9
The final act

LECTURE 9-1

Even after ten years my skills were still valuable and I soon found a job. I began picking up the pieces of my life...

If I know the function that relates distance and time, f, I can find the function that relates speed and time, g = ds/dt. If I know the function that relates speed and time, I can use Riemann sums and find the distance traveled over any interval of time during the motion. While this might seem like a significant accomplishment, I am not satisfied. This is the nature of my species; we are never satisfied and hence we have ‘progress’.

The burr under my saddle is giving distance traveled as an approximation process instead of an explicit number. I feel uncomfortable if I can’t give a number explicitly but only a way to approximate it.

I am not, however, consistent in this matter. The symbol, $\sqrt{2}$, also represents a number that can be given only as an approximation process and I use it without a second thought. If I am on a deserted island and am calculating with my fingers, toes, and a stick in the sand, how would I get a number for $\sqrt{2}$? The same way I would get a number for distance, I would approximate it. The truth is, $\sqrt{2}$ is no more accessible to me than the number the Riemann sums approach; I have to approximate both of them with finite decimals.

So why do I feel uncomfortable giving distance traveled as a number approached by Riemann sums but have no qualms at all saying that the distance between opposite corners of a square of side 1, is $\sqrt{2}$?

I think the difference is that I can express the length of the diagonal of a unit square by a symbol, $\sqrt{2}$. If I said that the length of the diagonal was the number that the products,

$$2(1-\frac{1}{16})(1-\frac{1}{16}\times\frac{1}{4})(1-\frac{1}{16}\times\frac{1}{9})\cdots(1-\frac{1}{16}\times\frac{1}{k^2}),$$

approached as k went to infinity, I don’t think I would feel so comfortable with it. The symbol, $\sqrt{2}$, hides the approximation process behind it.

I feel better about the $\sqrt{2}$ because it is a ‘symbol’ whereas the area under a curve or the distance an object travels has no symbol, just a finite decimal that is close to it.

I have been using the symbol, $\sqrt{2}$, for so long and it has become so familiar to me that I think of it as a Real World number. The picture of the diagonal of the unit square is such a compelling argument for the Real World existence of $\sqrt{2}$, that I am tempted to forget that I have never seen a rectangle, much less a unit square. I have only seen artists renditions of them.

Symbols seem to have an almost uncanny power to assume the mantle of reality, and the more I use one, the more I feel it represents something in the Real World. The substance behind the symbol, like the approximation process behind $\sqrt{2}$, is hidden and often complex. It is easy to forget about the substance and just deal with the symbol, which is in plain view and simple. If I am not careful, I can begin to believe the symbol is reality. Symbols are tricky.

I think symbols are devised in the Real World to represent Ideal World ideas and it is my personal opinion that there are no symbols in the Ideal World, just the reality of the Ideal World concepts themselves. Volumes have been written on the relationship between humans and their symbols and mathematical symbols are just another example. People have told me that they look upon a page of mathematics the same way they would look upon a page of obscure mystic runes. They don't realize that the page of mathematics is no more arcane than a page that holds the instructions on how to make bread. That may be a bad example.

The symbols of mathematics are different from other symbols representing abstract ideas in that they can be manipulated, which I think is a clever idea. The symbols, 1/2 and 1/3, represent Ideal World numbers and the form of the symbols is used to represent their product,
$1/2 \times 1/3 = 1/(2 \times 3)$. The symbols of mathematics are often quite ingenious.

I should point out that symbols can represent Real World ideas too, for example, finite decimals, if they aren't too long, and traffic signs. It is not that every symbol hides an approximation process and Real World symbols can represent Real World numbers that are exact. The symbol '2' seems pretty uncomplicated to me and I don't see much hidden behind it. I realize that this statement waves a red flag in front of some bulls, but '2' is certainly simpler than $\sqrt{2}$.

The symbol '9' represents the number of planets in the solar system which contains Earth as a member. I notice in passing that 9 can be divided into three equal parts. The expression, '1/3 of 9' makes Real World sense, so in this case, it appears the symbol, 1/3, acts like a Real World number. This seems at odds with the fact that 1/3 has an infinite decimal expansion and so should be an Ideal World number. Actually, I think that in this case the symbol, <u>1/3, doesn't represent a number at all</u>, but a <u>process</u> of dividing something into three equal parts. Sometimes it's a Real World process, sometimes an Ideal World process.

I think the symbols used for fractions are among the deepest in mathematics. The symbol '1/3' used in an expression like '1/3 of 7' can't stand alone, as a number, but must have something to divide into three equal parts. This process is modeled in the Ideal World by, '$1/3 \times 7$'. In the Ideal World, fractions are numbers that are ratios of integers and there is a list of rules that tells how to do arithmetic with them. End of story. I use one of those mindless rules and get 2 1/3 as the result of the process, which is an Ideal World number. This is an Ideal World process.

Now, in the Ideal World, the rules say that '$1/3 \times 7$' is the same number as '7 / 3', but '7 / 3' models something quite different. Seven divided by three is written as '7 / 3' or'$7 \div 3$', and models the process of repeated subtraction of one whole number from another until the amount remaining does not allow another subtraction. Now '7 divided by 3' gives me how many times I can take 3 away from 7 and how much remains when I can't subtract 3 anymore, so $7 \div 3 = 2$ with a remainder of 1; very Real World.

While the numbers, $1/3 \times 7$ and 7 / 3, are equal as Ideal World numbers, they model two different processes in two different worlds. I can't divide 7 into three equal parts in the Real World, but I can take 3 away from 7 two times with 1 left over. I think multiple use of symbols generally leads to some confusion.

So, I have decided that my uneasiness with distance or area follows from the fact that the numbers I get to represent them are not given in a way I am used to, namely as specific symbols. Well, no matter that symbols seduce me into false security, confuse me with their ambiguous multiple uses, and mystify me with ideas hidden behind their simple exterior, I am a human and my genes are what they are. I do not want my Riemann sums approaching a number in thin air, <u>I want them to approach a symbol</u>. And I want it now.

LECTURE 9-2

Picking up pieces is one thing, putting them together is another. They fit together in so many different ways...

I am hurting for a symbol that represents the number the Riemann sums of a function approximate. I could call it 'S' for sum or 'R' for Riemann or 'A' for area, but none of these give much information about what's going on. It would be nice if the symbol included the name of the function that models speed, say g, and the end points of the interval of time during which the object moves. With this information I can form the Riemann sums and approximate the distance.

A symbol that is used by some is

$$\int_a^b g\,.$$

The elongated 'S', $\int$, stands for 'sum'. This symbol gives me all I really need to find distance, but it does have some draw backs. It 'reads', "The sum of g from a to b", but I am not summing 'g' in a Riemann sum, I am summing g($t`_k$) times (t_k -t_{k-1}). I want to make the symbol read a little better.

It is a standard notation that the change in 'something' is denoted by Δ 'something'. If T stands for temperature, then I denote the change in temperature by Δ T. I can denote the change in time, (t_k -t_{k-1}), by Δt_k and write the Riemann sums as

$$\sum_{k=1}^{n} g(t`_k)\,\Delta t_k = g(t_1)\,\Delta t_1 + g(t_2)\,\Delta t_2 + ... + g(t_n)\,\Delta t_n\,.$$

This is nice but I don't want to represent a particular Riemann sum, I want to represent the number I get when all the "Δt_k 's have gone to zero", so to speak. I would like the symbol to somehow express this fact. I feel a twinge when I look at all the terms of the sum going to zero, and then I remember that even though the Δt_k 's go to zero, the sum doesn't go to zero because there are so many terms in the sum.

I now use a notation that goes back to the very origins of calculus. A Δ (quantity) that has gone to zero is represented by d(quantity). The Δt_k 's go to zero so I can represent the result of this by dt. I am summing g(t)dt over all the instants of time on the interval. Again using $\int$ as the symbol for summation, the symbol that I and almost everybody else uses for the number that the Riemann sums approach is

$$\int_a^b g(t)dt$$

This symbol 'reads', "The sum of g(t)dt over all the instants of time between t = a and t = b." I eventually want to interpret g(t)dt but before I do, I need to give some names.

This symbol represents a number and this number is called **the definite integral of g from a to b.** The process of going from the function that models speed to the distance traveled on an interval is called **integration**. The act of performing this process is the verb, **to integrate**. The numbers 'a' and 'b' are called the **limits of the definite integral** or **the limits of integration.**

If g is the function that models the speed of an object during the interval of time [a,b], then the distance traveled by the object in that time is represented by the definite integral

$$\int_a^b g(t)dt\ .$$

I have a symbol for the Riemann sums to approach and I really like this symbol because it tells me what is going on. The 'dt' speaks of Δt_k 's that have gone to zero in the service of the approximation of distance by the Riemann sums. This is one of the two legs upon which the definite integral stands. The symbol $\int$ reminds me that the definite integral depends on one of the tap roots of mathematics, the simple sum. This is the other leg.

This is my **'down home'** interpretation of $\int_a^b g(t)dt$. This interpretation says that the symbol is the number that Riemann sums approach. The 'a' and the 'b' give me the interval that I have to partition and the g(t) tells me the function to use in the Riemann sums. What more could I ask?

I like the Δ t, dt notation and as I think about it, average speed was a ratio of changes, and I should be able to incorporate the Δ idea into average speed. In fact, if I denote distance by s and time by t, then

average speed = change in distance/change in time = $\Delta s / \Delta t$.

This notation is suggestive more than explicit because it doesn't say what the interval of time actually is.

If I want to be explicit about average speed, I'll use the function that relates time and distance, s = f(t). Then,

(average speed over the interval from t_1 to t_2) = $f(t_2) - f(t_1) / (t_2 - t_1)$.

Now I have a change in f, $f(t_2) - f(t_1)$, and I can be suggestive again by writing

Average speed = $\Delta f / \Delta t$.

Speed is found by letting Δt go to zero which causes Δs, or Δf, to go to zero as well. I can use the 'dt', 'ds' notation and write speed as

$$\text{speed} = \lim \Delta s / \Delta t = ds/dt$$
$$= \lim \Delta f / \Delta t = df / dt,$$

where both limits are taken as Δt goes to zero.

The value of time where the speed is evaluated is not a part of ds/dt and this is a minor flaw in the notation.

I think this notation for speed is very suggestive. It <u>looks</u> like a ratio of changes. It reflects the idea that for instantaneous speed, the change in time has gone to zero since the ds and dt are a Δs and a Δt that have gone to zero.

It is tempting to think of dt as the length of an instant of time and ds the distance the object travels in that instant. This is treacherous ground. <u>The ds and dt are just symbols</u> and they don't have a numerical meaning. They remind me that ds/dt is the result of letting a Δt go to zero.

Be that as it may, I like thinking about dt as the length of an instant of time and ds as the distance traveled in that instant. Never let it be said that quicksand slowed me down. The <u>ratio</u> of distance to time has the 'feel' of instantaneous speed, and the very word, instantaneous, indicates that the 'time' in the ratio is the length of an instant of time.

The dt has length zero and yet the sum of the dt's over all the instants of time between 'a' and 'b' is b-a. I seem to be faced with a sum of zeros being greater than zero. Good. I like that. In the Ideal World, the dt's are a special kind of zero. If I add up enough of these special zeros, as many as there are instants of time in the interval, I get something that is non-zero. I need a special symbol for this super addition and the symbol is, $\int$.

I am not talking rigorous mathematics, I'm talking about how I <u>think</u> about the way the notation and symbols relate to the meaning of the integral and the derivative.

LECTURE 9-3

Inside each piece were seven pieces and inside each of these, seven more. I felt a sense of vertigo...

It is a pastime of mathematics to try to make 'things' behave as much like numbers as possible. It's like being a number is a very good thing to be, indeed, and 'things' ought to be very pleased if they are given some structure and the opportunity to be a little bit more like numbers.

I would like to do the honors for ds and dt and handle them as if they were numbers. They aren't numbers, but they act like numbers and I can pretend.

If g is the function that models speed, then $ds/dt = g(t)$. If I multiply through by dt and think of ds/dt as the ratio of ds by dt, I get

$$ds = g(t)\,dt.$$

I have used dt as though it was a number and I wonder if the result makes any kind of sense. Well, I can think of ds as the distance the object travels during the instant of time, t, and I can think of dt as the length of the instant. Since the speed does not change from one side of an instant to the other, the speed is constant during the instant of time t, and the motion is momentarily uniform. The speed is equal to g(t) during this instant of uniform motion, so the distance traveled is given by the product of g(t) and dt. Thus,

$$ds = g(t)\,dt,$$

and multiplying $ds/dt = g(t)$ through by dt gives a result that does make some kind of sense. Now I have an interpretation of g(t)dt in the definite integral,

$$\int_a^b g(t)dt\ .$$

During an instant of time, the speed is uniform and I can use the simple formulas,

$$\text{distance} = \text{speed} \times \text{time},\ ds = g(t)\,dt,$$

and

$$\text{speed} = \text{distance/ time},\ ds/dt = g(t).$$

I have returned to the two simple formulas for speed and distance that I had for uniform motion. The difference is in the size of the interval where the speed is uniform.

The total distance traveled during the interval of time, [a,b], must be the sum of the distances traveled, ds = g(t)dt, during all the instants of time between a and b. It is exactly the purpose of the symbol $\int_a^b$ to represent such a sum.

$$\text{(total distance traveled during the interval [a,b])} = \int_a^b ds = \int_a^b g(t)dt\,.$$

This is my **'exotic sum'** interpretation of $\int_a^b ds = \int_a^b g(t)dt$.

I 'read' $\int_a^b$ g(t)dt as

> "the sum of the (distance traveled in the instant of time, t)
> over all the instants of time between 'a' and 'b'",

where

> (distance traveled in the instant of time, t) = ds = speed $\times$ time = g(t)dt.

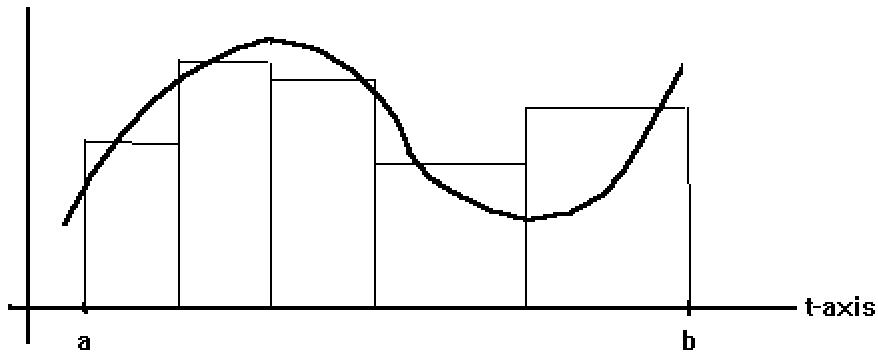
I have two interpretations of $\int_a^b$ g(t)dt:

$\int_a^b g(t)dt$ is a number that Riemann sums get close to. Riemann sums are finite, everyday kinds of sums and the number is attained through a rather prosaic approximation process. This interpretation is 'down home'.

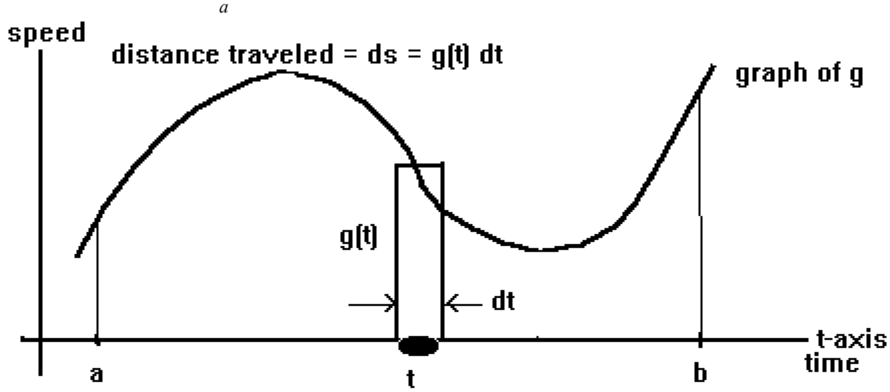
$\int_a^b g(t)dt$ is a number that equals the sum of all the distances traveled in all the instants of time between 'a' and 'b'. This interpretation is 'exotic'.

I think the symbol itself is more 'exotic sum' than 'down home' and when I look at $\int_a^b g(t)dt$, I see an 'exotic sum'. The exotic symbol, however, is just a glamorous facade that masks the Riemann sums inside doing all the work.

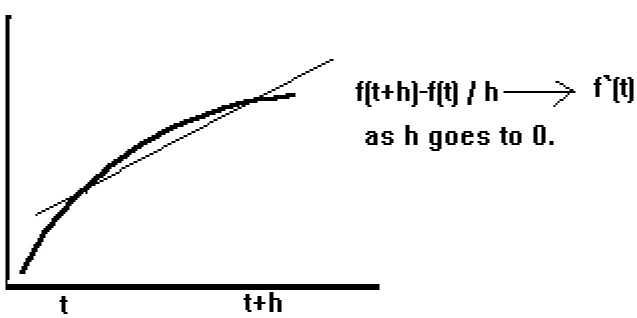
When I think of $\int_a^b g(t)dt$ in a 'down home' way, I think of this picture.



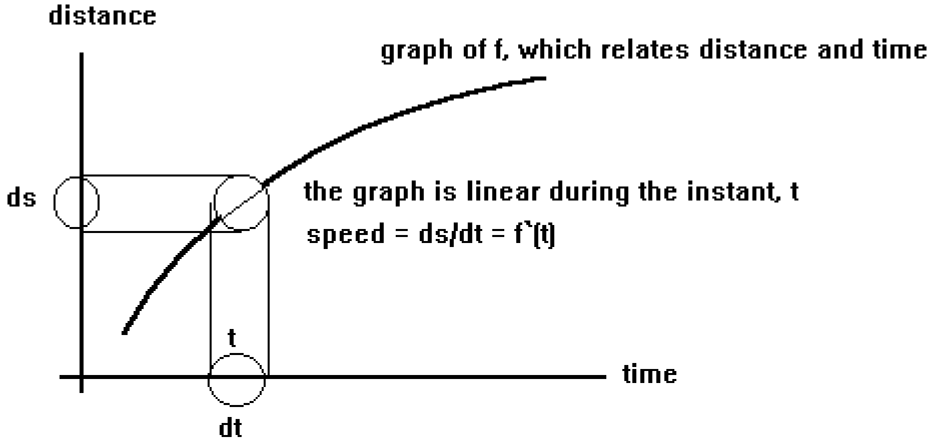
When I think of $\int_a^b g(t)dt$ as an 'exotic sum', I think of this



This is a 'down home' picture of the derivative.



This is an ‘exotic’ picture of the derivative..



LECTURE 9-4

I couldn't look at the pieces of my life, much less put them together. Pieces within pieces until they dwindled away to nothing...

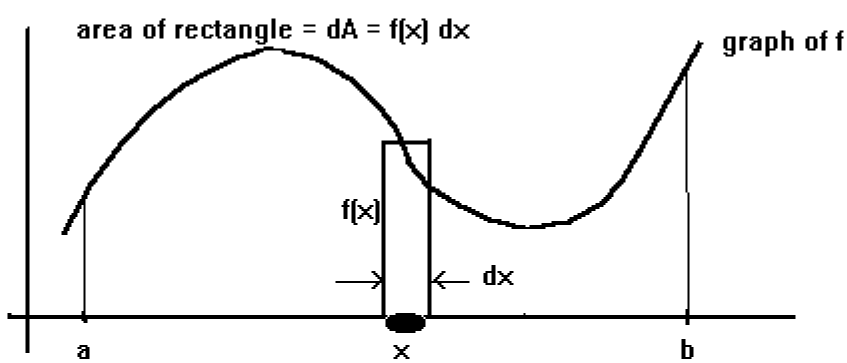
Any picture that I have of the definite integral brings area to mind. Area is the nut. No matter what the definite integral might represent, distance, for example, it can always be thought of as area. The distance is numerically equal to the area between the graph of the speed function and the horizontal axis.

A non-negative function may not model anything at all, but there can still be an area between its graph and the horizontal axis. The Riemann sums would approximate this area and the definite integral of the function would equal the area. If at some later time, I decided that the function modeled some physical quantity, say speed, I could interpret the area physically as say, distance.

I can think of area exotically. If x is a point in the interval, [a,b], then dx is the width of the point. The height of the graph, f(x), can't change between one side of x and the other, so I have a rectangle with height f(x) and base, dx. The area of this rectangle is f(x) dx. The area under the graph between a and b is the sum of the areas over all the points between a and b. I express this symbolically as

$$\text{area} = \int_a^b f(x)dx.$$

The picture looks like



There is nothing to prevent me from forming the Riemann sums of any non-negative function, even a function whose graph is undrawable and there is no obvious area between the undrawable graph and the horizontal axis. And who knows? Maybe the Riemann sums will approach some number as the mesh goes to zero. Maybe there is a number, L, such that given any error tolerance, τ, I can make all the Riemann sums within τ of L by making the mesh small enough. Maybe.

If a function that is defined on an interval [a,b] has Riemann sums that approach a number as the mesh goes to zero, then it possesses a property called **integrability** and the function is said to be **integrable**. I have quietly dropped the restriction that the function is non-negative for this definition. There is nothing in the definition of Riemann sum that requires that the function be non-negative and it is quite possible that the Riemann sums for a very general function could approach a number. I have generally resticted myself to non-negative functions because they are technically easier and I am interested in the ideas and not generality.

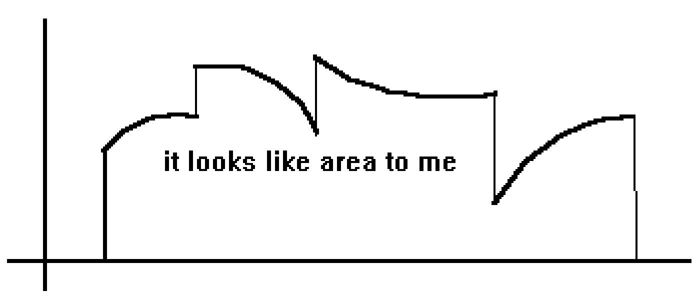
Any non-negative function whose graph, together with the horizontal axis, defines a region that has area, is integrable. This is, far and away, most of the functions I run into. Integrability is a property that is common among the functions I know, and is not at all special, but I know very few of <u>all</u> the functions there are. Most of the people I know are citizens of the United States of America, but most people are not. Is being integrable about as probable as being a Yank? Non-integrable functions are not mentioned much in polite conversation, but is it because there are so few that no one ever thinks of them, or is it because there are so many that they are commonplace? Or is it because there is some social stigma attached to non-integrability?

The property of being special requires that most of the objects in the population do not have that property. Without the non-special, there is no special. Both groups seem equal and necessary partners in the concept of 'specialty', but the special get names and the non-special don't.

I would like to find at least one non-integrable function because I like the idea that some are and some aren't. I have one friend who wishes all functions were and another who wishes all functions weren't.

I would also like to find an integrable function that has an undrawable graph. Not for any particular reason, just curious.

I can't think of any really bizarre, continuous, non-negative functions and I am going to assume that they are integrable. If a function has a finite number of jump discontinuities, it is integrable.



If I want an example of a function that isn't integrable, I'm going to have to look at functions with an infinite number of discontinuities.

I present a function, h, whose domain is [0,1], and whose rule is

h(x) = 1 if x is irrational
h(x) = 0 if x is rational.

The graph of this function is undrawable and it is discontinuous at every point in its domain.

Let P be any partition of [0,1], $0 = x_0 < x_1 < ... < x_{k-1} < x_k < ... < x_n = 1$. In every subinterval, $[x_{k-1}, x_k]$, there are a lot of irrational numbers, and I will choose one for a sample point $x`_k$ , when I form Riemann sums. With this choice, $h(x`_k) = 1$ for all values of k and

$$\sum_1^n h(x`_k)\Delta x_k = \sum_1^n 1\Delta x_k = 1 - 0 = 1.$$

No matter how small the mesh is, the Riemann sum equals 1. It would seem that the number 1 is a contender for the number that the Riemann sums get close to.

But every interval of the partition also has a lot of rational numbers. If I choose a rational number as a sample point, $h(x`_k) = 0$, and the Riemann sums are all zero.

$$\sum_1^n h(x`_k)\Delta x_k = \sum_1^n 0\Delta x_k = 0.$$

No matter how small the mesh is, there are Riemann sums that are equal to zero and Riemann sums that are equal to 1. The Riemann sums don't get close to a single number as the mesh goes to zero. The function h is not integrable.

The example gives me a feeling of what is involved in being integrable. As the mesh gets small, all of Riemann sums, and there are a lot of them, need to be near a number. In the example, the mesh doesn't control the Riemann sums.

If a function is continuous, it can't change much in a small subinterval. Because of this, the values of the function at the sample points are controlled which, in turn, controls the Riemann sums. In the example, the function took the values 0 and 1 in every subinterval and the mesh couldn't do anything about it.

The function, k, whose domain is [0,1], and whose rule is

$k(x) = 0$ if x is irrational
$k(x) = 1/q$ if x = the rational number p/q, reduced to lowest terms,

is continuous at the irrational numbers, discontinuous at the rational numbers, has an undrawable graph, and is integrable. None of these facts are entirely obvious.

These examples are deeply imbedded in the Ideal World. All functions that are Real World in the sense that they model the Real World, are integrable.

LECTURE 9-5

Since I couldn't think about the pieces of my life, I had time to think about other things. I started building a model airplane...

If an object is moving and if I know the function that models speed, g, then the distance traveled over any interval of time, [c , d], during the motion, is

$$\int_c^d g(t)dt.$$

The symbol tells me how to form the Riemann sums which approach the distance as the mesh goes to zero.

Now, I know the object exists and I know that it is moving, so there must be a distance function, f, which gives the distance the object has traveled at any time during the motion, and, in particular, at any time in [c , d]. I don't know what this function is, but I do know that its derivative equals g; f ` = g. The distance function is named for its ability to compute distance and I can use it to represent the distance traveled between t = c and t = d.

(distance traveled between t = c and t = d) = (distance traveled until time = d) -
(distance traveled until time = c)
= f(d) - f(c).

If I set the two ways to give distance traveled equal and use the fact that f ` = g, then I get the rather provocative result,

$$f(d) - f(c) = \int_c^d f`(t)dt .$$

I have been looking at the definite integral in a very physical way and have been thinking of the integrand as the physical idea of speed, not as the derivative of distance. The rock upon which I built the Riemann sums and definite integral, was the idea that

$$\text{distance} = \int_a^b (speed)dt .$$

I did not mention the function that modeled the distance traveled.

My arguments as to why the Riemann sums approach a number depend on the idea that the number is distance, and it is my intuitive feeling about speed, distance, and time that convinces me that the Riemann sums approach distance. I have called speed, g, not, f `, in these arguments, so while the derivative of f is present in spirit, in the integral,

$$f(d) - f(c) = \int_c^d g(t)dt,$$

it is ‘g’ that is there in the flesh and it is ‘g’ that I see, not ‘f `’. I am thinking of ‘g’ as the physical idea of speed, and not as the derivative of distance.

When I extended the idea of definite integral to non-negative functions whose graphs bound a region that has area, even the spirit of the derivative was gone. The integrand was not the derivative of anything, it was just a function whose ‘claim to fame’ was having a graph that bounded a region with area.

$$\text{Area} = \int_a^b (height)dx.$$

It was the visual impact of area and the approximating rectangles that supplied the intuition that convinced me the Riemann sums approached a number that represented area, and I didn’t consider if the height was the derivative of anything or not.

When I look at

$$f(d) - f(c) = \int_c^d f`(t)dt$$

however, I see the possibility of computing the integral by finding a function whose derivative is equal to speed. Since the derivative of the function that models distance traveled is the function that models speed, in retrospect it seems obvious that I should look for a function whose derivative is speed. I just didn’t think of it that way.

My reasoning would go like this:

The problem is to find the distanced traveled during the interval [c,d] if the speed is g, and I do this by computing $\int_c^d g(t)dt$. But I know that g(t) = f `(t) for some function, f, so I find this function, f, and evaluate it at t = c and t = d. The distance traveled would presumably be f(d) - f(c).

If I can evaluate the integral this way, I will not be thinking about distance when I do it, I’ll be thinking about the purely mathematical process of finding a function whose derivative equals a given function. I wouldn’t have to use Riemann sums. I wouldn’t even have to think about Riemann sums; I probably should think about them, but I wouldn’t have to.

I'll look at an example to see how this might work. I'll suppose that the speed of an object is given by $g(t) = 3t^2$ for $0 \leq t$, and I want to find how far the object moves between t = 1 and t = 3. This distance is equal to $\int_1^3 3t^2\,dt$.

$$[\text{Distance traveled between } t = 1 \text{ and } t = 3] = \int_1^3 g(t)dt$$
$$= \int_1^3 3t^2\,dt.$$

Now it just so happens that when I was computing derivatives, I happened to take the derivative of $f(t) = t^3$ and found that $f`(t) = 3t^2$. Fortunately I remember this, and I have stumbled across a function, f, whose derivative is equal to g. All I need to do now is evaluate f at t = 1 and t = 3.

$$\int_1^3 3t^2\,dt = \int_1^3 f`(t)dt = f(3) - f(1), \text{ where } f(t) = t^3,$$
$$= 3^3 - 1^3 = 26.$$

If this is correct, it surely seems a whole lot easier than computing Riemann sums. The problem is, I don't know if the answer is correct or not. I don't know if $f(t) = t^3$ is the distance traveled at time, t, or not, I just know that g is its anti-derivative of. It seems to me as though there might be a function, h, whose derivative equals g(t) and
$h(3) - h(1) \neq$ distance traveled. It is possible that f is such a function.

I could compute the Riemann sums and see if 26 is correct but instead I'll try to justify the method. I think that will be more fun.

If my example works, it works because of the curious circumstance that

$$\text{distance} = \int \text{speed } dt \ , \text{ speed} = \text{derivative of distance}$$

which gives

$$f(d) - f(c) = \int_c^d f`(t)dt.$$

LECTURE 9-6

Even if the pieces are infinitesimal, they do make up my life, which goes on from day to day. I do believe my life exists. After all, I do think....

Functions whose derivatives equal a given function seem to be the hot topic and they need a name. A function, G, where G\`=g, is called an **anti-derivative** of g. This definition has nothing to do with g being speed or anything else physical. It just has to do with functions.

If a function has an anti-derivative, then it has an infinite number of anti-derivatives. Indeed, since the rate of change of a constant function is zero, the rate of change of a function plus a constant is the same as the rate of change of the function. If G is an anti-derivative of g and C is a constant function, then G + C is an anti-derivative of g. This gives an infinite number of anti-derivatives of g, and for all I know, there could be more yet.

So one of the snags in using '$f(d) - f(c) = \int_c^d g(t)dt$' , where f is an anti-derivative of g, is that there are an infinite number of functions whose derivatives equal g and presumably I have to pick the right one, the one that gives the distance traveled.

I think I need for some more notation. I would like to have a way to write 'the derivative of (G + C)' other than (G + C) \`, which seems too weak a symbol for such a strong idea. There are times, and this is one, where I really want to <u>see</u> that I'm taking a derivative and the little 'primes', that could be random glitches in the copy machine, do not do the job. I do have another way to write the derivative of G, namely, dG/dt and I am going to extract the 'd/dt' from the symbol and use it for the <u>operation</u> of taking a derivative. The symbol, dG\dt, has the connotation of a change in G, dG, divided by a change in t, dt. d/dt(G) has the connotation of formally taking the derivative of G. dG/dt makes some attempt at keeping the Real World idea of rate of change in mind. d/dt(G) puts the formal, mathematical, Ideal World process of taking the derivative in the forefront.

Anyway,

$$d/dt(G + C) = d/dt\ G + d/dt\ C = d/dt\ G + 0 = d/dt\ G,$$

and g has a lot of anti-derivatives among which is the function, f, that gives the distance traveled.

If I am looking at a moving object, <u>I know that a function, f, that models the distance traveled exists and that its derivative equals the speed, g</u>. I know that g has at least one anti-derivative, f, and I have to determine what the other anti-derivatives are like.

There's a bunch of anti-derivatives of the form h = f + C and I wonder what such an h has to do with distance. Before I look at h, though, I should make sure I understand exactly what f is. I have said that f(t) gives the distance traveled at time, t, and I can think of f(t) as the reading on an Ideal World odometer at the time, t. The function, f, relates the odometer reading to time. But distance traveled from where? How long has this object been on the road? What highways and byways has it traveled?

I'm sure the story of the object's travels is truly fascinating but the first thing that interests me is f(c), the odometer reading at t = c. The distance traveled up until 'time = c', which is given by f(c), does not affect the distance traveled between c and d. This distance traveled depends only on the speed between c and d.

If I want to see how far it is from Deming to Lordsburg, I look at the odometer at Deming and subtract that reading from the reading in Lordsburg. The person driving the car behind me would do it the same way and get the same distance. It is very unlikely, however, that the readings on their odometer would be the same as mine; the difference of the odometer readings would be the same.

The function, h = f + C, also gives distance traveled but it has a different odometer which reads f(c) + C at t = c instead of f(c). If distance traveled depended on the odometer reading at t = c, it would be senseless to put them on cars.

In fact,

$$h(d) - h(c) = [f(d) + C] - [f(c) + C] = f(d) - f(c) + C - C = f(d) - f(c)$$
$$= \text{distance traveled.}$$

The odometer reading at t = c subtracted from the odometer reading at t = d is the distance traveled during the interval [c,d], regardless of which odometer I use.

I know that if f gives distance traveled, every function of the form, f + C , is an anti-derivative of g that also give the distance traveled, but the possibility still remains that there are anti-derivatives of g that do not give distance traveled. Does g have an anti-derivative, k, so uncivilized that k(d) - k(c) does not equal the distance traveled?

I'm going to see just how different two anti-derivatives can be.

LECTURE 9-7

And if my life exists, I should be able to integrate its separate, tiny parts, into a meaningful whole...

Suppose G and H are two anti-derivatives of g, a function that models speed. In order to see how different they are, I'll look at their difference, G- H. The derivative of their difference is zero,

$$d/dt(G - H) = dG/dt - dH/dt = g(t) - g(t) = 0.$$

What kind of functions have a zero derivative? Constant functions have a zero derivative. Are there any others? Is it possible for a non-constant F to satisfy F `(t) = 0 for all t in an interval [c,d]?

Well, I'll suppose that F is a function whose derivative is identically zero on [c,d]. Evidently the hypothesis includes that F is differentiable on [c,d] and, since the derivative is a constant function, it satisfies the 'principle of continuity'. If F isn't constant then there would be two points in the interval, u and v, where $F(u) \neq F(v)$. Then the average rate of change between u and v would be $F(u) - F(v) / u - v \neq 0$. But F is differentiable and the derivative satisfies the 'principle of continuity', so there is a time, c, between u and v when the instantaneous rate of change equals the average rate of change.

$$F`(c) = F(u) - F(v) / u - v \neq 0.$$

This can't happen because F `(t) = 0 for all times t, including t = c. F must be a constant function for otherwise I have absurdity. I can not argue with speed and average speed here because I don't know that F models distance traveled until after the argument.

The derivative of H - G is zero and this evidently implies that H - G = a constant.

If H and G are two anti-derivatives of g, they differ by at most a constant, and H = G + C, where C is a constant function. If f is any function that gives the distance traveled, then it is an anti-derivative of g, and <u>any</u> other anti-derivative of g must be equal to f + C for some constant, C. The function, h = f + C, also gives distance traveled but with a different odometer. <u>All of g's anti-derivatives give distance traveled</u>.

I can now put together a new way to find the distance an object travels.

There is an object in motion during the interval [c,d] and its speed is given by the function, g. The existence of the object in motion implies the existence of a function, f, that models the distance traveled and f ` = g. I do not have to worry about whether g has an anti-derivative or not, I know it does. If G is <u>any</u> anti-derivative of g, then

$$\text{(distance traveled between t = c and t = d)} = \int_c^d g(t)dt = G(b) - G(a).$$

I can evaluate the distance traveled, that is, I can evaluate the definite integral of speed, using any anti-derivative of g.

My computation,

$$\int_1^3 3t^2\, dt = 26$$

is correct. The function $f(t) = t^3$ is an anti-derivative of $g(t) = 3t^2$, and so works to find the distance traveled.

If g models speed, then I would seem to have a possible method to find distance that apparently doesn't use Riemann sums. It doesn't seem entirely unreasonable to expect that the method would work to find the definite integral of any non-negative function, g. The anti-derivative is a purely formal, mathematical concept, and is in no way interested in what, if anything the function models.

I am going to go through the argument that I used when g was speed and see how much of it I can do if g is any non-negative function.

If G and H are both anti-derivatives of g, then d/dt(G - H) = 0 as before. If d/dtF = 0, then F is a constant function using exactly the same argument that I used when g as speed. As a matter of fact, in the case where g was speed, I didn't know that F was distance traveled and I used a more or less visual argument with slopes of lines. Again I have that any two anti-derivatives can differ at most by a constant function.

If G and H are anti-derivatives of g, then G = H + C, where C is a constant function.

$$\int_c^d g(t)dt = G(d) - G(c) = [H(d) + C] - [H(c) - C],$$

$$= H(d) - H(c)$$

So if g has one anti-derivative that can be used to find the definite integral, any of them can be used.

I can show that this method works to evaluate the definite integral, $\int_c^d g(t)dt$, for a non-negative function, g, if I can find one anti-derivative, G, such that $\int_c^d g(t)dt = G(d) - G(c)$.

Now I have to see which non-negative functions have an anti-derivative that can be used to evaluate

$$\int_{c}^{d} g(t)dt.$$

Since continuous functions are the functions of applications, I'm going to look for an anti-derivative of a non-negative continuous function. Since the graph of a non-negative continuous function bounds a region with area, the Riemann sums get close to that area and the definite integral exists. The definite integral is no longer a distance but it is an area, and this interpretation works regardless of what the function models or whether it models anything or not.

LECTURE 9-8

I can easily see how fast my life is speeding by...

I am going to let g be a non-negative, continuous function, and I first want to find a function, G, such that

$$G(b) - G(a) = \text{Area under graph of } g = \int_a^b g(x)dx.$$

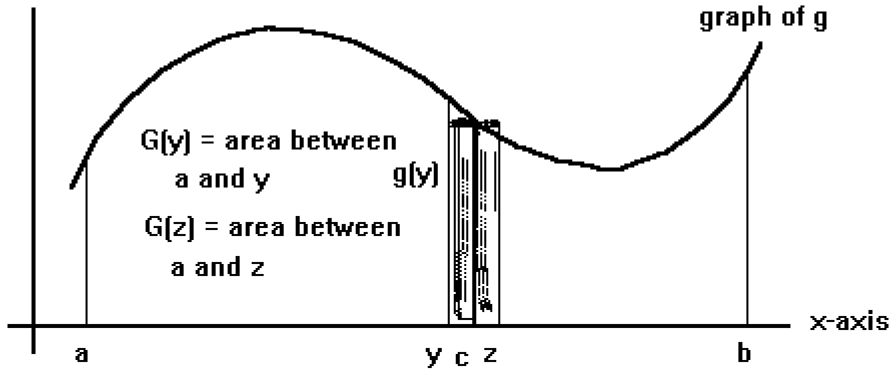
If I can find G, I will then try to show that G`(x) = g(x). In the case where G(b) - G(a) = distance traveled between t = a and t = b, the function G(t) gave the distance traveled from someplace until time, t. It seems reasonable that in this case, G(x) should give the area under the graph of g from someplace to the point, x.

Always trying to be reasonable, I will define G in this way. I only have to find one anti-derivative so I'm going to make it as easy as I can and let G be the function whose domain is [a,b] and G(x) is the area under the graph of g between 'a' and 'x'. This makes

$$G(a) = 0 \text{ and } G(b) = \int_a^b g(x)dx.$$

I let 'y' be any number between 'a' and 'b', a < y < b. It is my contention that G`(y) = g(y).

Well, G`(y) equals the limit of the difference quotients, G(z) - G(y) / z - y, so I'll see what I can do with them.



G(z)-G(y) = shaded area = g(c) (z - y) y<c<z

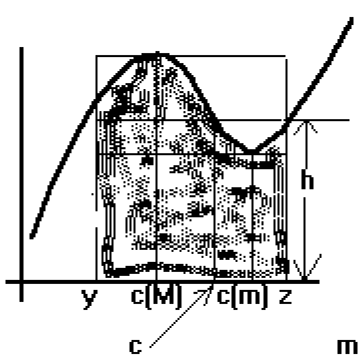
G(z) - G(y) = area of the region under the graph of g between y and z. Looking at the picture, I can see that this region is roughly rectangular with base (z - y). As a matter of fact, there is a value, c, between y and z, such that the area of the rectangle with base, (z - y), and height, g(c), actually equals the area of the region. The area of the region equals $g(c) \times (z - y)$ where $y \le c \le z$. This follows, as I will now show, from the 'principle of continuity'.

The area of the region is less than or equal to $g(c_M) \times (z - y)$ and greater than or equal to $g(c_m) \times (z - y)$, where $g(c_M)$ is the maximum and $g(c_m)$ is the minimum of g(t) in the interval [y,z]. By the 'principle of continuity', there is rectangle with base (z-y), whose area is exactly the area of the region, and whose height, h, lies between $g(c_M)$ and $g(c_m)$;

$$g(c_m) < h < g(c_M).$$

Since g is continuous, g(t) takes every value between $g(c_M)$ and $g(c_m)$ on the interval [y,z], and that means there is a value, c, in the interval [y,z], such that g(c) = h and the area of the region equals $g(c) \times (z - y)$.

I really feel better for having said that.



The base of every rectangle considered is (z-y). The largest rectangle has height, g[c(M)]. Its area g[c(M)]x(z-y), is larger than the area of the shaded region. The smallest rectangle has height, g[c(m)]. Its area, g[c(m)]x(z-y), is smaller than the area of the shaded region. By the 'principle of continuity' there is a rectangle between the two extreme rectangles whose area is equal to the area of the shaded region. The height of this rectangle is h and h must lie between g[c(M)] and g[c(m)]. g[c(m)] < h < g[c(M)]. Since g is continuous, it takes every value between g[c(m)] and g[c(M)], including h. Thus, there must be at least one number, c, where y < c < z and g[c] = h.

In particular,

$$G(z) - G(y) = g(c)\ (z-y), \text{ with } y < c < z, \text{ and}$$

$$G(z) - G(y) / z - y = g(c)\ (z - y) / (z - y) = g(c).$$

The difference quotient is, it seems, equal to g evaluated at some point x = c.
Since c is trapped between y and z, as z goes to y, c goes to y, and since g is continuous, $\lim_{c \to y} g(c) = g(y)$. The derivative is mine.

$$G`(y) = \lim_{z \to y} G(z) - G(y) / z - y = \lim_{z \to y} g(c) = \lim_{c \to y} g(c) = g(y).$$

Well, son of a gun, G`(y) = g(y). I find this extremely gratifying.

The function, G, is an anti-derivative of g and $G(b) - G(a) = G(b) = \int_a^b g(x)dx$. I can now use <u>any anti-derivative of g</u> to evaluate the definite integral.

<u>If g is a non-negative continuous function, then it has an anti-derivative G such that</u>

$$G(b) - G(a) = \int_a^b g(x)dx,$$

<u>and so, if H is any anti-derivative of g,</u>

$$H(b) - H(a) = \int_a^b g(x)dx.$$

The anti-derivative, G(x), that I know always exists, is the area under the graph of g between ‘a’ and ‘x’.

but integrating it seems much harder...

If g is a continuous, non-negative function then it has an anti-derivative, G(y), which is essentially an area under the graph of g from a to ‘y’. Since the anti-derivative, G(y), and $\int_a^y g(t)dt$ both represent the area under the graph of g between a and y, they are equal, and

$G(y) = \int_a^y g(t)dt$. This gives me the very suggestive relation which seems to say that differentiation undoes integration,

$$G`(y) = d/dy \int_a^y g(t)dt = g(y).$$

The symbol, $\int_a^y g(t)dt$, contains all the information necessary to compute the anti-derivative using Riemann sums. After all that work to avoid using Riemann sums, I am back to using Riemann sums.

On the other hand, taking derivatives can be reduced to a formal process that in its practice has little to do with limits or rates of change. The ideas of limits and rate of change is used to get some formal rules and using these rules, I can take the derivative of any function I can write down, giving no thought at all to limits or rate of change.

These formal rules can be found in every college calculus book written since 1936, which is as far back as I have checked.

The possibility exists that there are some formal rules for taking anti-derivatives and Riemann sums can be sidestepped after all.

The actual state of affairs is that there are formal rules for some functions and not for others. There are a variety of techniques for finding anti-derivatives, but they are far from completely successful. The class of functions whose anti-derivatives I can write down is distressingly small. There are enough of these functions to require a considerable amount of time to learn how to find their anti-derivatives, but many of the functions that come up in Real World problems do not have explicit anti-derivatives.

This situation leaves the Riemann sum and some variations on that theme with a big piece of the action. The high speed computer makes Riemann sums and similar but more refined methods quite attractive, actually.

I can not help but notice how area has usurped the basic notion of definite integral from distance. I tend to think of distance as the area under the graph of the speed function. Of course, the definite integral is neither distance nor area. The definite integral is the number the Riemann sums approach. Distance and area are Real World interpretations of the Ideal World definite integral.

How often the true meaning of something gets forgotten and is replaced by a convenient meaning. This may be unfortunate in some ways, but I am a Real World person and area means more to me than an abstract limit of Riemann sums. I think of the Riemann sums as approaching an area.

I have been assuming that g is non-negative. Speed is non-negative so when I was considering distance, the arbitrariness of this restriction was not noticeable. When I consider the definite integral of more general functions, continuity seems like a reasonable condition but non-negativity does not. As a matter of fact non-negativity is not an essential requirement. I kept non-negativity because it was easier. If I allow g to become negative I have a lot of technical details to worry about and I am more interested in the ideas, which are perfectly well illustrated using non-negative functions. Area below the axis is negative and to go through all the computations to put this into the mix, bores me. I am going to leap over this hurdle in much the same way that I jumped from increasing speed to any speed. I am going to be bold and just say it.

Continuous functions have anti-derivatives and if H is any anti-derivative of a continuous function, g, then

$$\mathbf{H(b) - H(a) = \int_a^b g(t)dt.}$$

My dream come true.

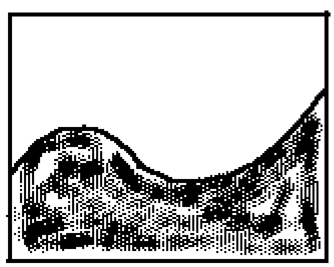
The actual computation of the anti-derivative could turn my dream into a nightmare.

There are many approaches to the computation of a definite integral, of which using anti-derivatives or Riemann sums are just two. Here are three more, all of which compute area.

1. If I draw the graph of the function on real World paper, and if I know the density of the paper in weight per square unit, I can cut out the area and weigh it. The weight multiplied by the density gives the area. I have been told that an analytical balance can weigh finger prints on a flask, so the method can be quite accurate.

2. I can draw the graph on finely lined graph paper and count the little squares in the region.

3. If I draw the graph in a rectangle of known area, A, and I throw darts randomly at the rectangle, then the ratio of the number of times that the dart hits the region to the total number of throws, should approach the ratio of the area of the region to the area of the rectangle. This method doesn't really require a graph, it requires some algorithm that allows one to determine if a dart hits the region or not.



number of hits / number of throws ⟶

area of shaded region / area of rectangle.

These kinds of methods are useful if only the graph of the function is known and not an explicit formula for its rule. Since many functions are known only by data obtained from experiments, these methods are applied quite often.

LECTURE 9-10

I flew my model airplane yesterday and it was caught by a thermal and disappeared into the clouds. I took that for a positive sign...

I will conclude with some examples.

I'll start with the simplest motion problem, an object moving with constant speed, say, $g(t) = 2$ for $0 \le t$. An anti-derivative of g is 2t so the general anti-derivative of g is $f(t) = 2t + C$. The distance traveled, s, is given by $s = f(t) = 2t + C$. I now have to decide about the constant, C.

When the motion is uniform, the graph of f, the function that models the distance traveled, is a line whose slope is equal to the speed. A line, however, is not determined by its slope. The slope determines an infinite family of parallel lines that differ by where they meet the vertical axis. If I think of f as relating an Ideal World odometer to time, then the speed determines an infinite number of such functions which differ by what the odometer reads at $t = 0$.

The family of lines, $y = 2x + b$, contains exactly the lines whose slope is 2. A particular line is picked out by giving the point where it meets the y-axis. There is only one line whose slope is 2 and passes through the point (0,4), and that is

$$y = 2x + 4.$$

I evaluated 'b' by substituting $x = 0$ and $y = 4$ into the equation and solving for 'b'.

$$y = 2x + b$$
$$4 = 2 \times 0 + b$$
$$b = 4.$$

The family of anti-derivatives, $f(t) = 2t + C$, contains exactly the functions that model the distance traveled if the speed is 2. A particular function is picked out by giving the odometer reading at $t = 0$. There is only one function that models the distance traveled whose odometer reads 4 when $t = 0$, $f(0)= 4$, and that is

$$f(t) = 2t + 4.$$

I evaluated C by letting $t = 0$, $4 = f(0) = 2 \times 0 + C = C$.

'Slope determines a family of parallel lines' is the geometric statement of the fact that a constant derivative determines an infinite family of linear functions, all of which differ by a constant. This extends to the fact that generally the derivative determines an infinite family of functions, all of which differ by a constant. One additional piece of information is needed to determine the function uniquely.

I'm not sure what the simplest non-uniform motion problem is but I am going to opt for a speed given by $g(t) = 3t^2 + 2t$ for $0 \le t$. I am not going to give an example of a physical object that actually moves with this speed, just assume there is one. I want to find the function, f, that gives the odometer reading at any time, t, $0 \le t$, if the odometer reads 6 when the clock is started.

My first approach will be to find the anti-derivative. Every differentiation formula gives an anti-differentiation formula so I look at derivatives to find an anti-derivative. The reason I chose this particular function for speed was that I knew that the derivative of t^2 is 2t and that the derivative of t^3 is $3t^2$. This gives me the family of anti-derivatives, $f(t) = t^3 + t^2 + C$, where C is any constant. I now find the value of C that gives the correct odometer reading at t = 0. the odometer reading at t = 0 is 6, so

$$f(0) = 6 = 0^3 + 0^2 + C = C.$$

The function that gives distance traveled as a function of time is $f(t) = t^3 + t^2 + 6$.

I can also look at this problem exotically.

The symbol, ds, represents the distance traveled during the instant of time, t. The constant speed during the instant of time, t, is $(3t^2 + 2t)$, and the distance traveled is the speed multiplied by the length of the interval of time, dt,

$$ds = (3t^2 + 2t)dt.$$

I can express the distance function, f(t), as the odometer reading at 0 plus the sum of all distances traveled during all the instants of time between 0 and t, that is, the definite integral of speed from 0 to t. I am running into a notational snag here. I want to represent the distance traveled between 0 and t as a definite integral and the upper limit of the integral would be t. But t is also the symbol I'm using for the general instant of time in the integral, and

$$\int_0^t g(t)dt$$

doesn't make sense. I am going to keep the 't' in the upper limit because I want to use 't' as the independent variable of the distance function, and use a different symbol for the general instant of time in the integral.

f(t) = distance traveled at time, t.

f(t) = (odometer reading at t = 0) + (distance traveled between times 0 and t).

$$f(t) = f(0) + \int_0^t g(\tau)d\tau.$$

$$f(t) = 6 + \int_0^t (3\tau^2 + 2\tau)d\tau = 6 + (t^3 + t^2) - (0^3 + 0^2) = 6 + t^3 + t^2.$$

The answer, of course, is the same; the way of looking at it is different.

It does not really occur to me to use Riemann sums. I only think of Riemann sums if I can't find an anti-derivative.