CHAPTER 6

A line by any other name would smell as sweet.

LECTURE 6-1

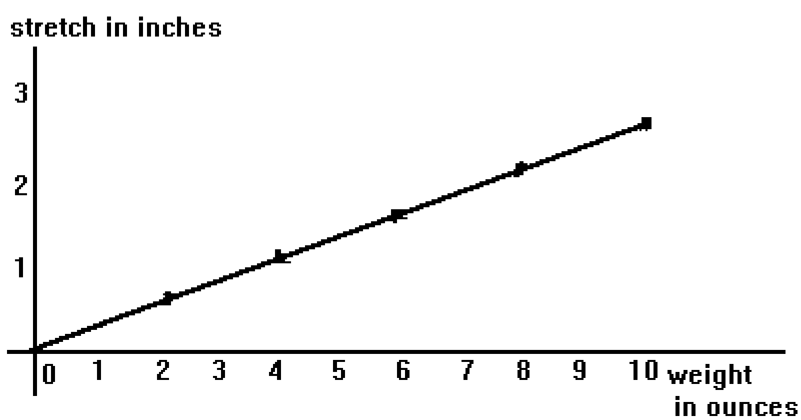
I and my suitcase were unceremoniously put out of the bus by the side of a long straight road that vanished into the horizon in both directions...

In the Real World I perform experiments and take data. For example, I hang various weights from a spring and see how far the spring stretches with each weight.

weight ounces	distance stretched inches
0	0.00
2	0.51
4	0.97
6	1.51
8	2.03
10	2.48

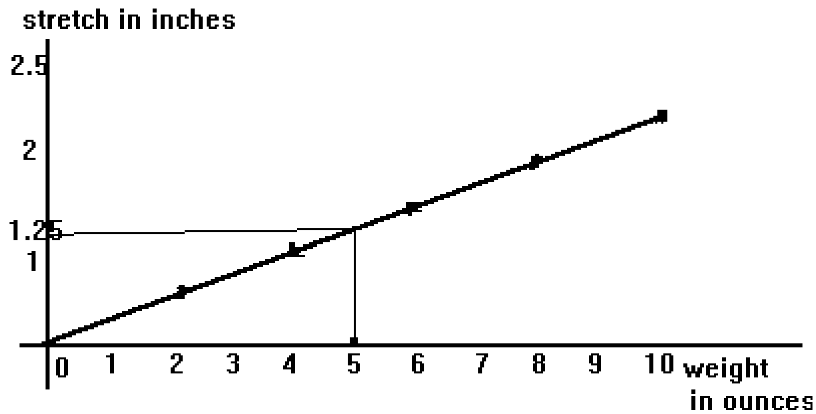
There are a couple of things I can do with this data. I can see if I can find a pattern in it and come up with a formula that relates the weight to the stretch. Eventually I would hope to use the formula as the rule of a function that models this experiment. I can also plot the data in a coordinate system and see what that tells me. I am going to do the latter first.

I draw a horizontal and vertical axis, decide on a scale, and put the weights on the horizontal axis and the stretch on the vertical axis. I am anticipating that the weight will be the independent variable of a modeling function and the stretch, the dependent variable. I made this choice is because I think of the weight causing the spring to stretch.



I plot the six points, (w,s), where w is the weight and s is the stretch. The points look as if they might be on a line so I put a ruler along them and, sure enough, the edge of the ruler goes through all of them. I draw the line.

I drew the line for more than esthetic reasons. I want to use the line to find the stretch for any weight up to 10 ounces. If I want to find the stretch from a 5 ounce weight, I find 5 ounces on the w-axis and the stretch ought to be the height of the line above that point.

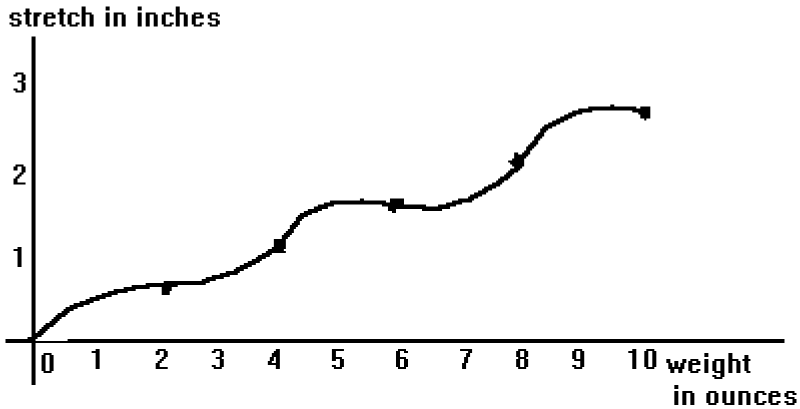


The height of the line looks to be about 1.25 inches. The only way to check this is to hang a 5 ounce weight on the spring and see what happens. Well, I did that and measured the stretch. I got 1.24 inches, which is close enough for me. The fact is, that I have hung a lot of weights from a lot of springs and the data points always lie on a line.

This, as usual, is not entirely true. If I put so much weight on the spring that it is stretched out of shape, the points don't lie on a line anymore, but I am not at the moment interested in over-stretched springs. If a point (*a,b*) lies on the line, then it is my opinion that *a* ounces will stretch the spring *b* inches. This is a Real World decision based on Real World experience.

I am totally in the Real World. I have been using Real World numbers, Real World data points and have drawn a Real World line. A Real World point is a dot of ink or graphite on a piece of paper. A **Real World line** is a mark I draw on a piece of paper using a ruler and pen or pencil. In the Real World, points lie on a line if they all lie along the edge of a ruler. My justification for drawing the line through the six points and using the line to compute stretches for different weights is Real World experience.

If this had been my first experiment, I would have been less sure that the line was correct. It might be that the curve I should use to compute the stretches looks like



but experience tells me it is a line and not this....serpent.

I can use the Real World data to try to make a formula that relates weight and stretch. If I round the data off to the nearest tenth, my try is met with success and I get

$$s = 1/4\ w.$$

The 1/4 represents the amount of stretch for an ounce of weight. This is exactly analogous to the distance traveled in a unit of time, which came up when I was looking at uniform motion. In that case I called the number speed, but in this case I have no name for the 1/4. A child without a name.

I can only be sure that the formula works for six values of weight but the same experience that convinces me that all the data points lie on a line, convinces me that the formula works for all weights up to 10 ounces.

This is very reminiscent of my point of view about the graph of the function that modeled uniform motion which was also a line. Now I want to figure out exactly what a line is.

LECTURE 6-2

I sat on my suitcase. Looking down the road in one direction was the same as looking down it in the other. Had I not remembered that cars drove on the left, I could not have told which direction I was going nor from which direction I came...

A friend of mine had been fooling around with electricity and she told me that she had found a formula that relates the voltage across a piece of iron to the current that passes through it. I did not, at the time, know much about electricity, but not wanting to rain on her parade, I wrote the formula down and said that I would look at it.

She let ‘V’ represent voltage in volts and ‘I’ represent current in amperes. She said that there was a number R such that

$$V = R \times I.$$

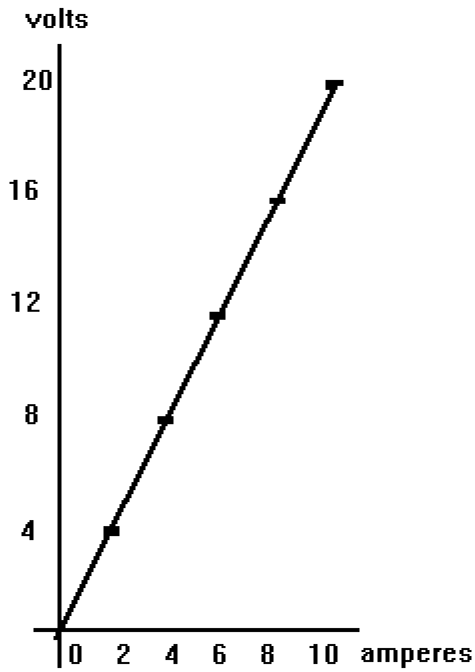
She called the number, R, resistance. I presumed she had her reasons for doing so.

When I returned home I got curious about this voltage and current stuff. The formula was a lot like the formulas I had obtained in my study of uniform motion and the spring. The resistance seemed to be, at least formally, like speed and the unnamed quantity, inches of stretch per ounce of weight. I supposed that resistance must be the number of volts per ampere of current, although I didn’t really know what those terms meant.

I thought that I would draw the picture of this formula so I made a table using a value of R = 2. Remembering my previous efforts, I evaluated the voltage at equally spaced values of current.

current in amperes	voltage in volts
0.0	0.0
2.0	4.0
4.0	8.0
6.0	12.0
8.0	16.0
10.0	20.0

I drew a vertical and horizontal axis, chose a scale, and plotted the points.



I put a ruler along the points and saw that they fell along its edge. The picture of this formula was a line. Actually, I wasn't too surprised since I had experienced this kind of thing before.

There is a difference between this example and the previous example. In the first example I was in at the beginning. I took the data, I plotted the points, I saw that the picture was a line, and I found the formula that related weight and stretch. In the last example, I was given the formula, used it to plot points, and saw that the points lay on a line. It seems to me that I should be able to just look at the formula and see if its picture is going to be a line or not. If I could do that, then I could just plot two points and save myself the trouble of making the table.

At the moment, I am working mostly with formulas but I am thinking of them as rules of functions that model a particular physical system. The lines will then be the graphs of these functions. My aim is to be able to tell from the rule if the graph of a function is a line.

LECTURE 6-3

But did they drive on the left? I had been in so many countries lately and that now I wasn't so sure....

The lines I drew as pictures of the weight-stretch relation and the voltage-current relation are really Real World line segments and I would model them in the Ideal World as segments of Ideal World lines.

The lines I drew as pictures of the coordinate axes are Real World pictures of Ideal World lines. These are pictures of lines that go on forever.

The mark I see on the paper could be either of two pictures. A line drawn on a coordinate system could be a Real World line segment that pictures some physical process and is going to be modeled by a segment of an Ideal World line or, it could be a Real World picture of an infinite, Ideal World line. Context tells me which.

A finite number of points lie on a Real World line segment, or are **collinear,** if whenever I put a ruler on two of them, the rest of the points lie along the ruler as well. This is how I tell if points on my paper are collinear. I use a ruler.

This definition of collinear doesn't work in the Ideal World because I can't get there to use the ruler. In the Ideal World, a line is an ideal mathematical object, and it truly consists of Ideal World points. If I want to show that a point lies on an Ideal World line, I need some kind of criterion so that if the point satisfies the criterion, then I know the point lies on the line. The Real World 'points lie on the edge of a ruler' criterion is denied me in the Ideal World.

In order to see what this criterion might be, I'm going to have to decide what an Ideal World line is. It would seem impossible to show that a set of points is on a line if I don't know what a line is. I need a definition of 'line' in the Ideal World.

Someone might object that since I have already used lines for asymptotes and in forming the axes of coordinate systems, it is a little late to make a big point about the definition of a line.

What I have done is assume that the concept of line is part of common knowledge. One aspect of this common knowledge is that I need two points and a ruler to draw a line. In the Real World I need a Real World ruler made of metal or plastic. In the Ideal World there is the ideal ruler that is used in the constructions of classical Euclidean geometry.

Part of the common knowledge of lines is that a segment of a line is the shortest distance between two points. Evidently, I am also assuming some common knowledge about distance and what it means. As a matter of fact, I am going to assume that anything I need to know about the lines of Euclidean geometry is common knowledge. As a matter of fact, I am going to assume that anything I need to know about anything, and don't want to develop in detail, is common knowledge. I am assuming that all the words I am using are part of common knowledge. I am assuming that numbers are part of common knowledge, and I am assuming that standard mathematical notation is part of common knowledge, to mention just a few of the assumptions that I'm making. I do not apologize for them.

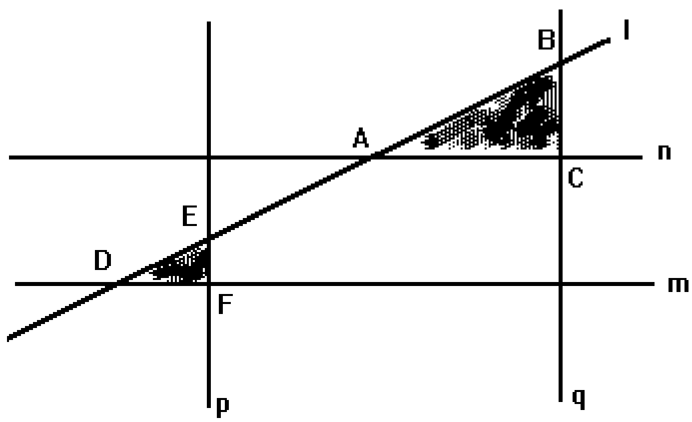
My plan is to put an Ideal World line from Euclidean geometry in an Ideal World coordinate system and then find some way to describe the line in terms of the coordinates of the points that lie on the line. This means that I will have to transfer geometric knowledge of the line to analytic knowledge of the relation between the coordinates of the points that lie on the line.

This last statement implies that I have geometric knowledge of lines to transfer. If I am unsure of the geometric facts about lines, I would not find any argument using these facts very convincing. It is hard to understand things about lines that are based on things about geometry that I don't understand.

Fortunately, I think I have a pretty good shot at understanding the geometric ideas I need because I can <u>see</u> the geometric ideas and more often than not, they <u>look</u> right. By 'understanding an idea' I mean that I believe the idea is true and I see how it makes sense with respect to the rest of geometry that I know.

The equation, '$2x + 3y = 6$' does not 'look' like a line and the equations, '$2x + 3y = 6$' and '$2x + 3y = 9$' do not 'look' like parallel lines. It is hard to visualize these lines and the algebraic approach to lines does not appeal much to my intuition. Even if my knowledge of the axioms and theorems of Euclidean geometry is spotty, I feel some confidence in being able to comprehend the objects themselves.

Here is a geometric fact that I want to believe. In the picture below, the line **l** is a transversal of the parallel lines **n** and **m**. The lines **p** and **q** are both perpendicular to both of the lines **n** and **m**. I want to believe that the shaded triangles are similar. **Triangles are similar if their corresponding angles have the same number of degrees; or if the ratios of corresponding sides are equal; or if the triangles are scale models of each other**.



I could argue that since corresponding angles formed by two parallel lines and a transversal have the same number of degrees, the angles at A and D have the same number of degrees. Further, because the lines **p** and **q** are perpendicular to both of the lines **m** and **n**, the angles at C and F both have 90 degrees. Finally, the sum of the degrees in the angles of a triangle is 180 degrees, so the angles at E and B both have the same number of degrees. So, if I believe the stuff about transversals of parallel lines, that right angles have 90 degrees, and that every triangle has 180 degrees worth of angle, which I do, then I believe the triangles are similar.

But what if I didn't believe the geometric facts that seem to imply that the triangles are similar? Well, similar also means that the triangles are scale models of each other and the shaded triangles look that way. The corresponding angles of the triangles look as though they have the same number of degrees.

I may not understand the geometric argument, but I can believe that the triangles are similar because they look similar. Their similarity feels right to me. In the privacy of my thoughts, I might even admit that my *belief* in the similarity of the triangles was due more to how they looked than the geometric proof.

Arguing Ideal World facts on the basis of Real World pictures is not a generally accepted practice but I try to rise above that prejudice.

LECTURE 6-4

Thinking that 'right was might', I turned to my right and began walking. I had been facing the road when I turned to the right. Had I been on the other side of the road, I would be walking the other way. Was I egocentric enough to think that I was on the 'right' side of the road. I stopped...

I am going to think of a line lying in a coordinate system in the Ideal World. The line doesn't model anything in the Real World so I will call the horizontal axis, the x-axis and the vertical axis, the y-axis.

Each point on the line has coordinates and I want to find some numerical relation that these coordinates satisfy. I would like the numerical relation to be an equation since, in my book, an equation is about simplest numerical relation there is. Further, if a point is not on the line, then its coordinates should not satisfy the relation. I want this relation to pick out all the points that lie on the line and only those.

I'm going to look at a few special cases first. My hope is that the relationship between the coordinates will be simpler in these cases and I can have a little early success while getting a feel for the problem. With this in mind, the first cases I consider are the horizontal line and the vertical line.

A line is horizontal if it is parallel to the x-axis. If a horizontal line hits the y-axis at the point b, then the y-coordinate of every point on the line is b. It is also true that if a point is not on the line, then its y-coordinate is not b. This is a very negative way of saying that if the y-coordinate of a point is b, then it lies on the line.

Since the y-coordinate of every point on the line is the same, any point on the line supplies the common y-coordinate of all points on the line.

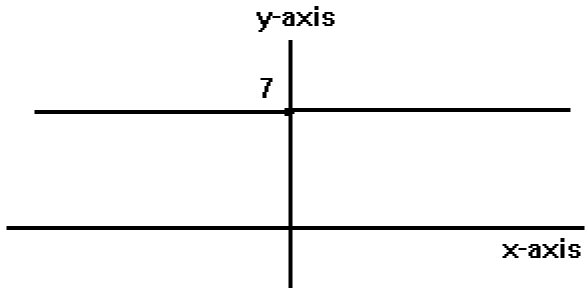
If the point (a,b) lies on a horizontal line, then a point (x,y) lies on the line if and only if y = b.

If (x,y) is on the line, then y = b. If y = b, then (x,y) is on the line.

The equation that determines all the points on the line is 'y = b' and the <u>name</u> I give the line is the equation that determines which points lie on it. For example, 'y = 6' and 'y = π' are names of lines. When I say, "...the line 'y = -2'...", I mean the set of all points whose y-coordinate is -2. The x-coordinate of the points on the line is not restricted at all.

There is some ambiguity here. 'y = 2' could be a line or a point on the vertical axis. Usually context makes it clear which is meant. If the context does not make it clear, I should say, "...the line y = 2 or the point y = 2 ", which I do unless I forget.

Horizontal lines are graphs of functions since they pass the vertical line test. Let g be the function whose domain is all the real numbers and whose rule is g(x) = 7. Then the graph of g is the line 'y = 7'. Here is the rather unexciting graph of g.



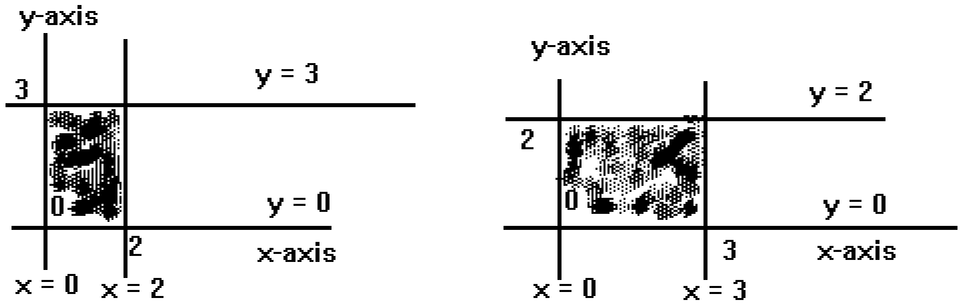
This function models Real World situations. For example, I noticed my old dog, Spot, lying 7 feet from the Old Oak Tree. I made the Old Oak Tree the zero point of distance and made the starting of my clock the zero point of time . For all I knew, Spot had been lying there for days. Knowing Spot, it was likely. I watched Spot until dinner time and he never moved. If y is the dependent variable and stands for distance and x is the independent variable and stands for time, then g models Spot's distance from the tree as a function of time.

Spot has not has not been lying there since time began; indeed, I saw him at the backdoor last summer, and Spot will not be in that place forever for a variety of reasons. He has been lying there since sometime in the indefinite past and will lie there into the indefinite future so, while the Real World domain of g is not the whole real line and its graph is really only part of the Ideal World line, y = 7, I will probably model it as the whole line.

A line is vertical if it is parallel to the y-axis. The equation that describes the points on a vertical line is similar to the equation that describes the points on a horizontal line, but it involves the other coordinate. Every point on a vertical line has the same x-coordinate.

If the point (a,b) lies on a vertical line, then the point (x,y) lies on the line if and only if x = a, and 'x = a' is the name I give to the line.

A vertical line can't be the graph of a function since it fails the vertical line test in spades. Still, the vertical line is useful. I can model a 2 ft. by 3 ft. piece of Real World sheet steel by a rectangular region in a coordinate plane. I would model it as the region bounded by the lines x = 0, x = 2, y = 0, and y = 3, or by the lines x = 0, x = 3, y = 0, y = 2, depending on how I wanted to position it.



I have already used vertical and horizontal lines for asymptotes and I named them in the same way as I have here. The difference is that then it was *ad hoc* and now it is part of a general scheme that I hope will name all lines.

LECTURE 6-5

I crossed to the other side of the road to see if it felt like the 'right' side, but I didn't notice any difference. After a little experimentation I found that I could not remember which side I started on...

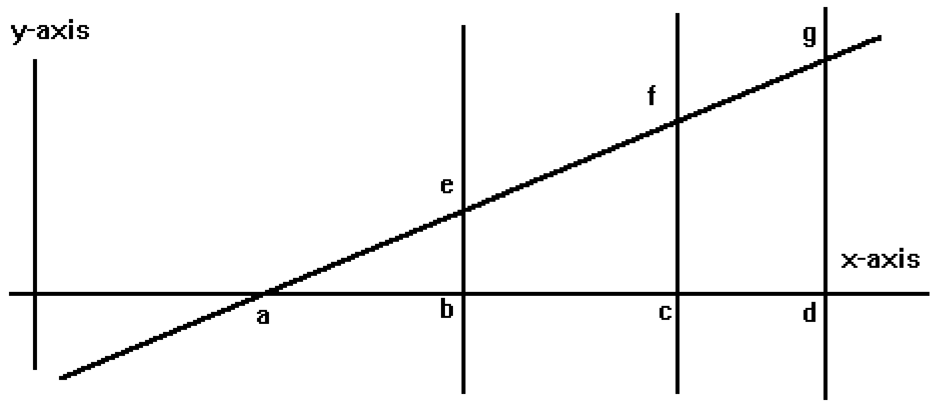
Now I'm going to look at a line that is parallel to neither axis and so is neither horizontal nor vertical. To begin with I am going to think of the line as rising; as the x-coordinates of points on the line get bigger, so do the y-coordinates. I am eventually going to have to consider lines that fall as I move to the right, but I don't want to get bogged down with cases in my first effort.

Traditionally two points determine a line in both worlds, but it is not immediately clear how to get from two points on the line to an equation that the coordinates of all the points on the line must satisfy. I need an intermediate idea that gives me a bridge between the two points and the equation.

The intermediate idea is the direction of the line. If I know a point on the line and the direction of the line, then the line is completely determined. If certain information completely determines a line, then I should be able to get the equation that the coordinates of the points on the line must satisfy from that information. Of course the equation must be inherent in the knowledge of two points on the line also, but it is a little more accessible from the knowledge of a point and a direction.

In the Real World I can give the direction of a line by giving the acute angle that the line makes with the x-axis. Since the line is rising, this angle will be between 0 and 90 degrees. I can also give the direction by pointing and saying, "That way.", but this is a little hard to model in the Ideal World.

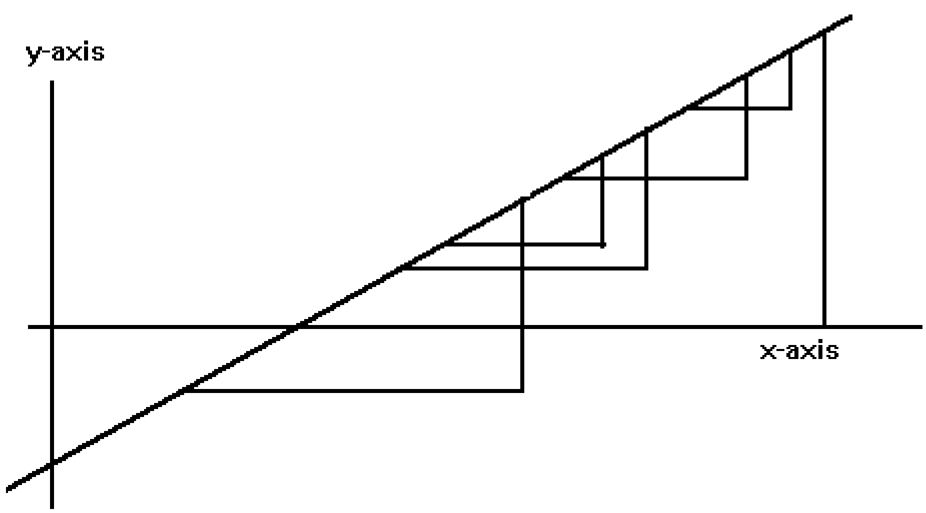
I am going to use the angle between the line and the x-axis but I am going to use it in a geometric way as opposed to a trigonometric way. The basic idea is that <u>all the right triangles formed by the line, the x-axis and a vertical line are similar</u>. This means that the ratio of the vertical side of the triangle to the horizontal side is the same for all of these triangles.



I am going to let <u>ab</u> represent the distance between the points a and b, <u>ac</u> represent the distance between the points a and c, and so on. The triangles abe, acf, and adg are all similar and

<u>eb</u> / <u>ab</u> = <u>fc</u> / <u>ac</u> = <u>gd</u> / <u>ad</u> .

As a matter of fact, **all of the right triangles that have legs parallel to the axes and hypotenuses on the line are similar**. I'm going to call these triangles, **magic triangles**.

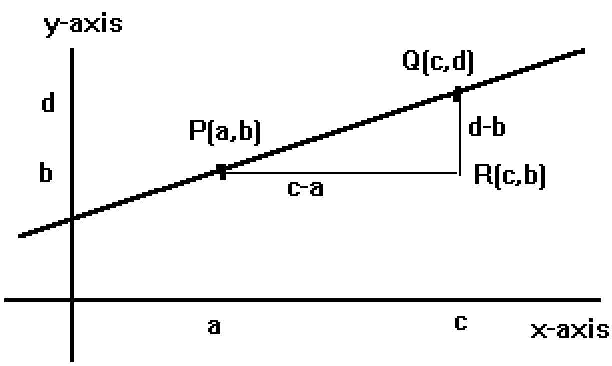


These triangles are all similar because of a theorem of geometry that I talked about earlier. I am not going to refer to that theorem, however, but will justify my statement that all of these triangles are all similar by assuming that their similarity is common knowledge. I defend this assumption on the basis that all the triangles look similar. The small triangles look like scale models of the big triangles and the other way around. I put the triangles all on one side of the line because that is the way I think of them. I could have put them on the other side or on both sides; they are all similar.

Since these triangles are all similar, the ratio of the vertical leg to the horizontal leg is the same for all of them. **This common ratio is the number that carries the direction information of the line.** If I have any two points on the line, I can find this ratio.

This omnipresent ratio is called the **slope** of the line. If the vertical side of a magic triangle is a lot bigger than the horizontal side, then the slope is large and the line slopes a lot. If the horizontal side is a lot longer than the vertical side, the slope is close to zero and the line doesn't slope very much. The line is almost flat or horizontal.

Since this ratio is the same for any of the magic triangles, I can use any of the triangles to compute it.



I am going to call the two points on the line P(a,b) and Q(c,d), using the notation that both names the points and gives their coordinates. If I call the point where the horizontal and vertical sides of the triangle meet, R(c,b), then the triangle PQR is one of the magic right triangles. Length is an inherently non-negative quantity and it is important that the formulas I use for lengths give non-negative numbers. I put the point, Q(c,d), to the right of P(a,b), so c > a and (c-a) > 0. Since the line is rising, d > b and (d-b) > 0.

The ratio of the vertical side to the horizontal side, the slope of the line, is

$$\text{slope} = (d-b)/(c-a).$$

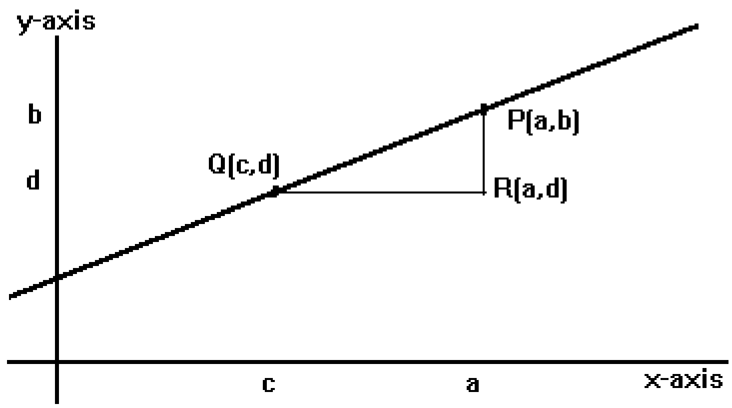
If P(a,b) and Q(c,d) are any two points on the line, then the slope is given by

slope = (c - a) / (d - b) .

If (a,b) is any point on the line and (x,y) is any other point on the line, then the slope is given by

slope = (y - b) / (x - a).

Perhaps I am a little over eager. I have only looked at the case where the point Q lies to the right of the point P. Maybe I should try Q to the left of P. There might be a sign change.



If Q is any point on the line other than P, I don’t know if it is to the left or the right of P but I would hope the same formula

slope = (d - b) /(c - a)

works.

I’ll put Q(c,d) to the let of P(a,b). In this case, keeping the lengths of the sides of the triangle positive, the slope is given by (b - d) /(a - c) , the ratio of two positive numbers.

But (c-a) is the negative of the length of the horizontal side and (d-b) is the negative of the length of the vertical side and the situation is saved by the algebraic oddity that a negative number divided by a negative number is a positive number. The formula

slope = (d - b) /(c - a)

gives the correct positive number.

Formally, the computation looks like this

slope = length of the vertical side / length of the horizontal side

= (b-d) /(a-c) = (-1) (d-b) / (-1) (c-a) = (d-b) / (c-a) = slope formula.

Here again I have used some algebra; common factors in the numerator and the denominator can be canceled.

The upshot of all this is that if P(a,b) and Q(c,d) are any two distinct points on a line, then

slope = (d-b) / (c-a) =(b-d) / (a-c).

It doesn't make any difference whether I subtract the coordinates of P from the coordinates of Q or the coordinates of Q from the coordinates of P, I just have to be consistent. I can't subtract the x-coordinate of Q from the x-coordinate of P and the y-coordinate of P from the y-coordinate of Q, for example.

Going back to the ranch, if (a,b) is any point on the line and (x,y) is any other point on the line

slope = (y - b) / (x - a) .

The point (x,y) must be another point, because if (x,y) = (a,b), then the formula would give 0/0. Using 0/0 is a social gaff of the first order in any world I know of.

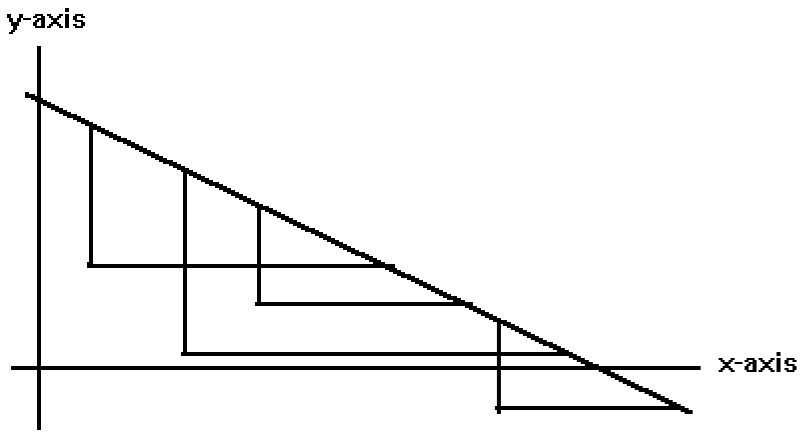
I note that the slope is a positive number for the lines I have considered, lines which are rising.

LECTURE 6-6

I was in an isolated part of the country and it was possible that I could go fifty miles in either direction and not find habitation. Asking which direction had the closest town seemed a meaningless question...

I have a number that describes direction for half of the lines, the rising lines, and the other half, the falling lines, are next. The y-coordinates of points on falling lines decrease as the x-coordinates increase.

I still have a family of similar triangles. I generally put them under the falling line and with no more reason than I put them to the right of rising lines.

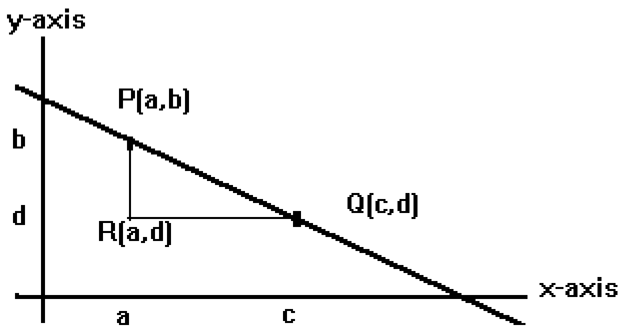


The ratio of the length of the vertical side to the horizontal side is the same for all of these triangles and I am tempted to call this ratio the slope of the line as I did before. The problem is that length is a positive number so that the slope of a falling line would be positive and slope would not differentiate between a rising and falling line. This would seem to be a serious deficiency in the definition.

While the basic definition of the slope of a rising line,

length of vertical side / length of horizontal side

doesn't work, I also have a formula for slope in terms of the coordinates of two points on the line. The nature of the formula seems to be 'sign sensitive' and I'm going to see what happens when I use the formula on the falling line.



I let P(a,b) be any point on the line and Q(c,d) any other point on the line. Again I have exercised my insensitive bias toward the right and put Q to the right of P in the picture. Now my formula for slope is

$$\text{slope} = (d - b) / (c - a).$$

Since Q is to the right of P, (c-a) is positive, but since the line is falling, $d < b$, and (d-b) is
negative. The ratio of a positive number and a negative number is a negative number. The absolute value of the ratio is the ratio of the lengths of the sides, but the ratio itself is negative.

I am going to drop the geometric 'ratio of lengths of sides' as my definition of slope and use the formula.

There are a couple of rationales for this.

First, I am trying to describe the properties of a line with respect to a coordinate system, so it seems reasonable to define the slope in terms of coordinates and not strictly in geometric terms.

Secondly, mathematics lives and dies by its rules. Time and again mathematics extends ideas by making definitions that force the old rules to work in new situations.

The rules for exponents is a case in point. Originally exponents were used as a shorthand for repeated multiplication in much the same way that multiplication by a positive integer is shorthand for repeated addition.

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2.$$

The rules for exponents come from this idea, in particular,

$$a^n \times a^m = a^{n+m}$$

I n order to preserve this rule, a^{-n} was <u>defined</u> as

$$a^{-n} = 1 / a^n \ .$$

Multiplying 2 times itself a negative 3 times makes no sense at all to me but I can use 2^{-3} formally in computations because it is defined in such a way as to make the rule work.

$$2^5 \times 1/2^3 = 2^5 \times 2^{-3} = 2^{5-3} = 2^2 \ .$$

This agrees with my ideas of cancellation since

$$2^5 \times 1/2^3 = 2 \times 2 \times 2 \times 2 \times 2 / 2 \times 2 \times 2 = 2 \times 2 = 2^2 \ .$$

I use 2^{-3} in expressions when I am using the rules because that is the form of $1/2^3$ that works best with the rules. 2^{-3} doesn't mean much to me intuitively, so when I see it, I automatically and immediately change it in my mind to $1/2^3$ = 1/8 which I do have some intuition for.

I am now ready to formally define the slope of a rising, falling, horizontal or vertical lines.

Let **l** be any line that is not vertical and let (a,b) and (c,d) be any two points on the line. The ratio

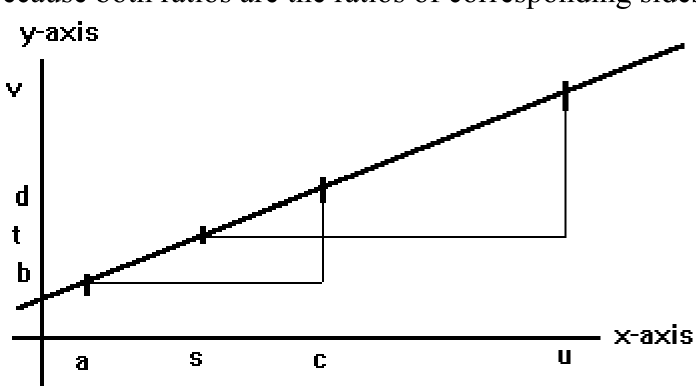
$$(d - b) / (c - a)$$

is called the **slope** of the line.

The definition does not depend on which two points I pick. If (u,v) and (s,t) are another pair of points on the line, **l**, then

$$(d - b) / (c - a) = (v - t) / (u - s)$$

because both ratios are the ratios of corresponding sides of similar triangles.

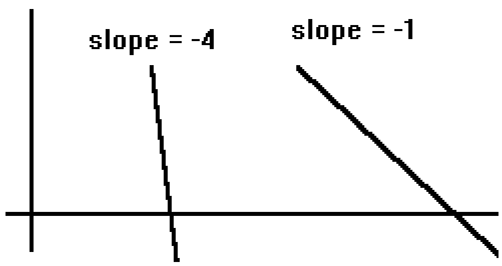


I have chosen the points in the picture so that all the differences are positive but it would work out OK no matter how I picked them. Trust me.
As long as I subtract the coordinates of one point from the coordinates of the other point, the order in which I do it doesn't make any difference.

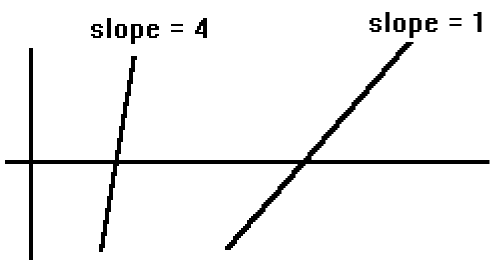
$$(t - v) / (s - u) = (-1) (v - t) / (-1) (u - s) = (v - t) / (u - s).$$

<u>If the slope of a line is positive, it is rising, and if the slope of a line is negative, it is falling.</u>

A line with a slope of -4 falls faster than a line with a slope of -1.



A line with a slope of 4 rises faster than a line whose slope is 1.



The line y = 3 is a horizontal line. Let (a,3) and (b,3) be any two points on this line. Then its slope is

$$\text{slope} = (3 - 3) / (b - a) = 0 / (b - a) = 0.$$

The slope of a horizontal line is zero. If the slope of a line is zero, the slope is neither positive nor negative, and the line is neither rising nor falling. It makes sense that if the slope of a line is zero, the line is flat or horizontal.

If I try the formula on a vertical line, say x = 3, I run into trouble. Let (3,c) and (3,d) be two different points on the line. If I naively use the formula I get

$$\text{slope} = (c - d) / (3 - 3) = (d - c) / 0$$

which is undefined.

My first reaction is a bit of a shock. Here is a whole family of lines for which the slope, the magic number, is not defined. Vertical lines don't have a slope. Perhaps I will have to scrap the whole idea and try to find something that works for all lines.

When I consider the situation a little more calmly, I recall that there other places where a problem with definition arises.

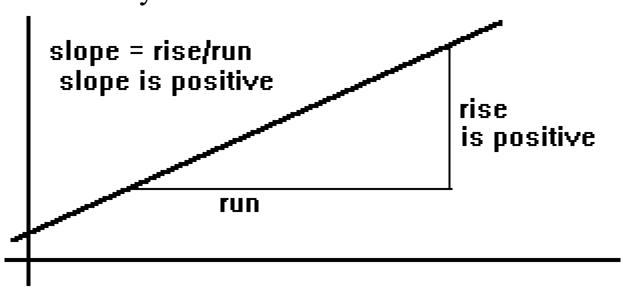
The reason why the slope of a vertical line is not defined is that division by zero is not defined in arithmetic. I live without being able to divide by zero, so I can live without slopes for vertical lines.

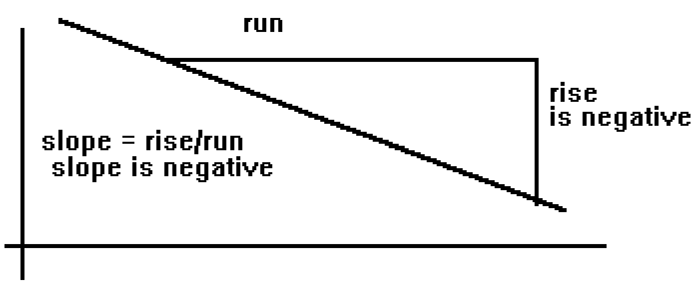
Having no slope is not the same as having a slope of zero. Zero is as good a number as any and perhaps better than most. Horizontal lines have a slope of zero. A vertical line just flat doesn't have a slope. This, in a backhand way, defines the slope of a vertical line. If I am told that a line doesn't have a slope, then I know the line is vertical.

It is commonly said that slope is 'rise over run', that is,

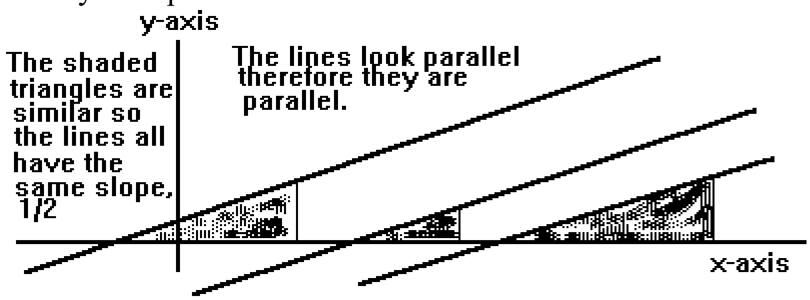
slope = rise / run.

Run is the change in the x-coordinate between two points and is usually thought of as being positive. Rise is the corresponding change in the y-coordinate and is positive if the line is rising and negative if the line is falling. I am using the usual meaning of negative as the opposite of positive. I often use the expression 'rise over run' for slope because I like the way it sounds.





There is one more observation I need to make. **Lines that have the same slope are parallel.** This is a result of Euclidean geometry but I am going to put it into common knowledge. Lines with the same slope meet the x-axis at the same angle and this would seem to imply that they are parallel. I am convinced by the fact that they look parallel. At least they look parallel to me.



LECTURE 6-7

It finally occurred to me that the presence of the road was narrowing my view of the possibilities. I wasn't limited to the two directions provided by the road, I could walk in any direction. The horizon seemed flat and identical in all directions but I began examining it with care...

How much information does it take to determine a line? If I know that the line doesn't have a slope, then I have narrowed it down to an infinite number of vertical, parallel lines. If I know that the line does have a slope, m, then I have again narrowed it down to an infinite number of parallel lines. Knowledge of the slope of a line narrows down the possibilities of which line it might be, but doesn't pin it down exactly. I need more information.

I have the direction, I need the point. There is only one line that has a slope of two and passes through the point (1,2). Given any point in the plane, there is a line of slope m that passes through it and there is only one line of slope m that passes through it. Given any point in the plane there is exactly one vertical line through it. As I think about this, it seems remarkable. Of all the lines of slope m, each point has its own. Of course, they have to share but there is nothing inherently wrong with sharing.

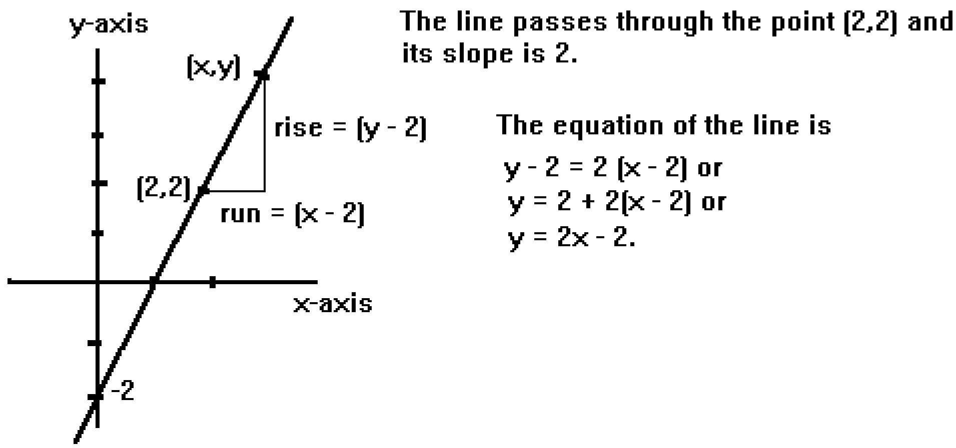
I can now write the equation that the coordinates of points on a line must satisfy. A line has slope m and passes through the point (a,b). If (x,y) is any other point on the line then

$$(y - b) / (x - a) = m.$$

This is the equation I seek. This is almost the equation I seek. There is one point on the line that does not satisfy this equation, the point (a,b). The problem is again division by zero. Since the problem is in the denominator, I'll get rid of it by multiplying both sides of the equation by (x - a).

The coordinates of every point on the line satisfy the equation

$$(y - b) = m (x - a).$$



I am going to say, "... a point satisfies the equation..." instead of, "...the coordinates of a point satisfy the equation...". This is how shop talk comes into being; impatience. I get in a hurry and want to get on with the program so I shorten what I have to say.

There is a little question about the equation that is gnawing at me. If I have a line, **l**, that passes through the point (a,b) and has slope m, I have convinced myself that every point on **l** satisfies the equation

$$y - b = m\,(x - a).$$

But does every point that satisfies the equation lie on the line. Could there be a point (u,v) such that $v - b = m\,(u - a)$, but (u,v) doesn't lie on the line? I hope not and I think I better look into the possibility. The method in my madness is to get (u,v) on some line and then convince myself that this line is the same as the given line. This will get (u,v) on the given line.

The two points, (a,b) and (u,v), determine a line, **n** , and

the slope of **n** = $(v-b) / (u-a)$.

But, I am assuming that $(v - b) = m\,(u - a)$, so I also have

$$m = (v - b) / (u - a)$$

and m is the slope of **n**.

But the slope of **l =** m so **n** and **l** have the same slope and must be parallel. Since they share the point (a,b) they must be the same line and **l** = **n**. Finally, (u,v) lies on **n** so it must lie on **l**.

The set of points that satisfy the equation $(y - b) = m\,(x - a)$ is exactly the line with slope m that passes through the point (a,b). This is exactly the relation I was looking for. Every point on the line satisfies the equation and no others do.

I'm going to name the line with slope m that passes through the point (a,b), "y - b = m (x - a)" but I will usually call "y - b = m (x - a)" the **equation of the line** and not the 'name of the line'. If I want to talk about the line with slope m that passes through the point (a,b), I will talk about the line "y - b = m (x - a)".

There is a minor inconvenience with this equation. I'm going to look at the line y = 2(x-3). This is the line that passes through the point (3,0) and has slope = 2. But the point (4,2) satisfies the equation and so lies on the line. The equation of the line with slope =2 that passes through the point (4,2) has the equation (y - 2) = 2 (x - 4). This line has the same slope as y = 2(x-3) and the lines share the point (4,2). They are the same line.

The line apparently has two names, y = 2(x-3) and (y - 2) = 2 (x - 4). From this point of view, the line has an infinite number of names or equations because there are an infinite number of points on the line to use for (a,b).

This multiple name phenominum is a quirk of algebra and there are many ways to write the same algebraic expression. For example, (x - 2) (x - 3) is the same as $x^2 - 5x + 6$, even though they look quite different.

It is a remarkable property of the names I have given lines that I can do algebraic operations on them. Since the names are equations, I can do anything to the names that I can do to equations.

If I do a little algebra,

$$y = 2(x-3)$$
$$y = 2x - 6$$

and

$$(y - 2) = 2 (x - 4).$$
$$y - 2 = 2x - 8$$
$$y = 2x - 6$$

I see that if I write the equation in the form $y = \alpha x + \beta$, the name is unique. At least in these two cases the name is the same and I believe that it will always be the same, no matter which point I use for (a,b). The number α in the expression $y = \alpha x + \beta$ is evidently the slope. If x = 0, then $y = \beta$ and the point $(0,\beta)$ lies on the line. But $(0,\beta)$ also lies on the y-axis so this is the point where the line intersects the y-axis. The point $(0,\beta)$ is called the **y-intercept of the line**. A line has only one slope and only one point where it hits the y-axis, so no matter what name I start with for the line, it will always end up with the same equation, y = (slope) (x) + (y intercept).

It is more or less standard to write y = mx + b, where m is the slope and b is the y-intercept. This is called the **slope-intercept** form of a line. "y - b = m (x - a)" is called the **point-slope** form of a line. Everything has to have a name.

How do I tell if two different looking equations are the same**. Two equations are called equivalent if the same numbers satisfy both of them.** In the case of the equations that are the names of lines, they are equivalent if exactly the same points satisfy both of them. So the names of two lines are equivalent if and only if they name the same line.

I can multiply both sides of an equation by a non-zero number and get an equivalent equation and I can add a number to both sides of an equation and get an equivalent equation. This gives a lot of possible names for one line.

Any equation that is equivalent to an equation of the form

$$ax + by = c, \text{ where at least one of a or b is not zero}$$

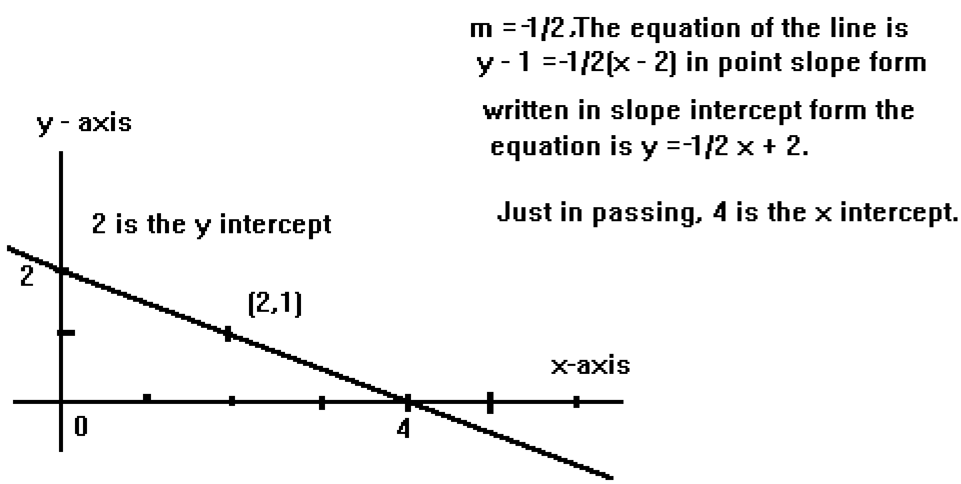
is a line. If a is not zero and b = 0, then it is the equation of the vertical line x = c/a. If b is not zero and a = 0, then it is the equation of the horizontal line y = c/b. If neither a nor b is zero, then

I can add (-ax) to both sides to get $by = -ax + c$
I can multiply both sides by 1/b to get $y = -(a/b)\, x + c/b$

which is the slope-intercept form of a line with slope = -(a/b) and y intercept = c/b. So ax + by = c is equivalent to the slope-intercept form of a line. If I were compulsive I could continue and write the equation as

$$y - (c/b) = -(a/b)\,(x - 0)$$

and see that ax + by = c is equivalent to the point-slope form of the line. Fortunately I will resist that temptation.



I usually use the point-slope form to write the equation of a line since the information I usually have is a point on the line and its slope. Other people might have other information and use one of the other forms. I change the algebraic form of the line as the spirit moves me.

LECTURE 6-8

Finally, I thought I saw something that might be a mountain in the far distance and set off toward it. Several difficult days later I was drinking from a cool stream at the base of the mountain and eating mangos. I saw another mountain in the distance and set out for it...

I want to revisit some of the previous examples and I'll start with the spring. In this example I plotted data on a coordinate system and decided that the data points were on a line because they fell along the edge of a ruler. I eventually wound up with a formula that related the weight in ounces on the spring, w, and the amount it stretched in inches, s, s = 1/4 w. I can use this formula to define a function, f ,that models the spring in the Ideal World. The domain of f is the interval [0,10] and the rule is f(w) = s = 1/4 w.

The graph of f is the set of all points (w,s) where

$$(w,s) = (w, f(s)) = (w, 1/4\ w)\ \ ;\ 0 \le w \le 10.$$

I am hardly amazed that the graph of f is part of the set of all points that satisfy the equation

$$s = 1/4\ w,$$

which is the equation of a line. The slope of the line is 1/4 and its s-intercept is the origin.

In another example, f models an object in motion. The domain is the interval [0,20] and the rule is s = f(t) = 2t where s represents distance in feet and t represents time in seconds. The graph of f is part of the set of points, (t,s), that satisfies the equation, s = 2t, which is the equation of a line with slope 2 and whose s-intercept is the origin. Again, my surprise is minimal.

Slope is a critical aspect of a line, its physical significance probably deserves some consideration.

In the first example, the units of rise are inches and the units of run are ounces. The units of rise / run are inches / ounce. There is 1/4 inch of stretch for every ounce of weight.

In the second example, the units of rise are feet and the units of run are seconds. The units of rise / run are feet / second. The object moves 2 feet of distance for every second of time.

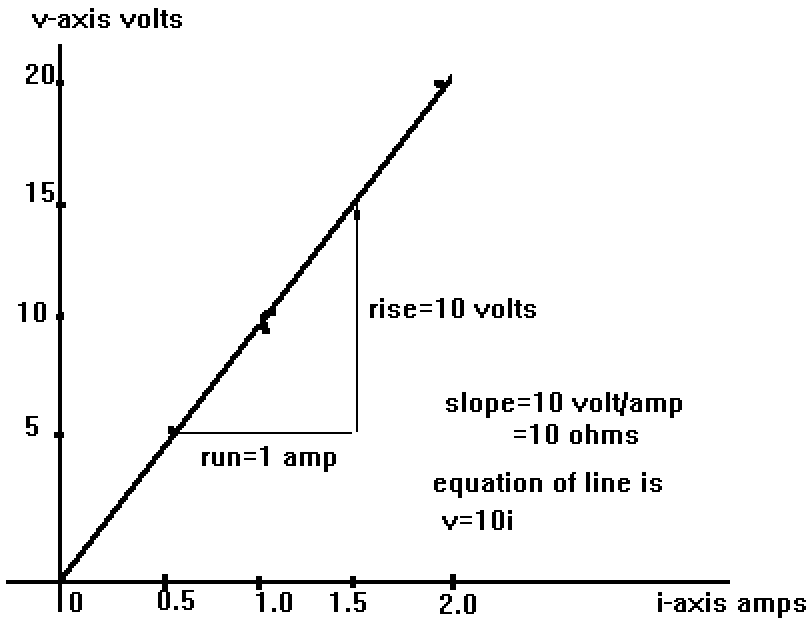
In each case the slope represented the change in one quantity for a unit change in the other quantity. Slope is computed by dividing (any change in the quantity represented by the dependent variable) by (the change in the independent variable that brought it about).

If the line doesn't model anything, rise is the change in y and run is the change in x. Slope is the change in y brought about by a unit change in x. Slope is a ratio of changes or a rate of change. I think of **slope** as a **rate of change**.

I would now handle the results of some of the experiments differently, particularly the experiment with the spring and the experiment with voltage and current. In these experiments data was taken and used to come up with a rule for the function that modeled the physics. Now I would plot the data and when I saw that the data points were on a line, I would find the equation of the line and use this equation to get the rule. I wouldn't have to find the rule in the data.

Suppose I put measured amounts of current, in amperes, through a piece of some material and then measure the voltage, in volts, across the material.

current amps	voltage volts
0.00	0.00
0.50	5.10
1.00	9.95
1.50	14.97
2.00	20.06



I eyeball a line through the data points. I am quite sure that (0,0) is on the line and I make sure that it is on the eyeballed line. I can choose any rise-run triangle to find the slope and I use a ruler to draw the triangle. I use the horizontal and vertical axis to measure the sides of the triangle and I choose the run to be an easy number that isn't too big or too small, like 1. Finally, I divide the length of the vertical side by the length of the horizontal side to find the slope, which in this case is what my friend called resistance.

The equation of the line is $v = 10i$. Voltage as a function of current is modeled by the function V whose domain is [0,2] and whose rule is $v = V(i) = 10i$. The slope is called the resistance of the material and the unit is the **ohm**, named after a man by the same name.

This experiment could be used to find the resistance of a material.

LECTURE 6-9

About half way between the two mountains I stopped to rest and fell asleep in the shade of a large boulder. When I awoke I could not distinguish one mountain from the other. The cloudy days and unfamiliar night skies made the decision more difficult. Even if it didn't make any difference which mountain I chose, I still had to...

I now come to one of the most important ideas of mathematics; the idea of **linearity**. If the graph of a function is a line, the function is said to be **linear**. A function models a **linear physical process** if the function is linear.

Suppose that f is a linear function and its graph is the line, y = mx + b. The graph of f is the set of points, {(x , f(x))} and the line is the set of points { (x , y) | y = mx + b }. Since these two sets are the same, it must be that f(x) = mx + b. Evidently, the rule of every linear function is of the form, f(x) = mx + b.

Every non-vertical line passes the vertical line test and so is the graph of a function. If ax + by = c , $b \neq 0$ is the equation of a non-vertical line, the signed distance of the line from the point, x, on the axis is y = -(a/b) x + c/b. This means that the line is the graph of the function whose rule is f(x) = -(a/b) x + c/b. Every line is the graph of a linear function.

When I look at a line, I see not only a line, I see the graph of a linear function. When I see a linear function, I see not only the function, I see the line that is its graph.

Generally a physical process is not linear but it is often pretty close to linear on part of the domain of the function that models it. Interest is usually centered on that part of the physical process where the graph of the modeling function it is pretty much like a line.

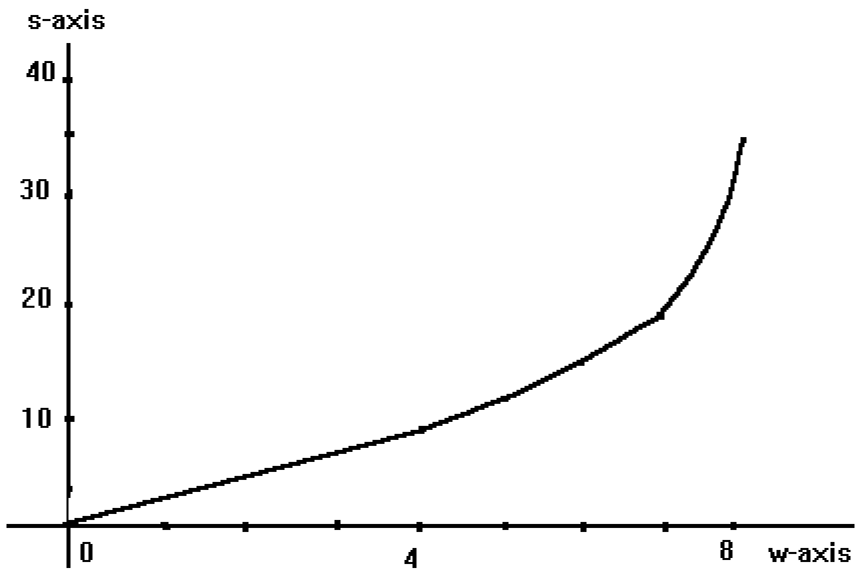
Specifically, there is some line y = mx + b and a part of the domain of the function where the graph of the function and the line are indistinguishable. In terms of functions, y = f(x) = mx + b for x in that special part of the domain. No matter how complicated the actual rule may be, on this particular part of the domain, the rule is particularly simple.

A spring is an example of this. When I start stretching a spring, the amount of stretch, s , is proportional to the weight ,w, that stretches it. That is, s = kw. If I take the weight off, the spring returns to its original shape. If, however, I put enough weight on the spring, it doesn't go back to its original shape, it deforms. A weight is reached where the metal yields. When the weight gets close to the yield value, I start getting more stretch for an added unit of weight. A little bit of weight of gives a lot of stretch.

The data might look like this:

weight ounces	stretch inches
0.0	0.0
1.0	2.0
2.0	4.0
3.0	6.0
4.0	8.0
5.0	11.0
6.0	15.0
7.0	21.0
8.0	35.0

If I plot the data and follow the dots it looks something like this:

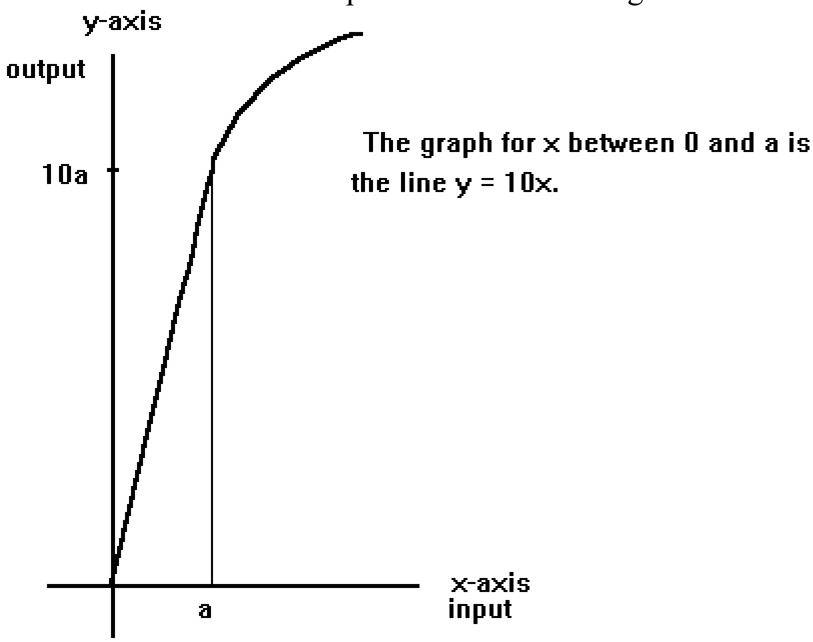


There is a function g whose domain is [0,8], whose rule is unknown to me, and whose graph looks pretty much like the picture I got from the data. There is a function f whose domain is [0,4], whose rule is f(w) = 2w, and whose graph looks just like the data picture for values of w between 0 and 4.

If I am going to use this spring in some mechanical system, I use the linear function f to model it and make sure that I don't load the spring with more than 4 ounces. If the spring is going to see more than 4 ounces, I have to use a different spring, one that is linear up to the largest weight it sees.

Non-linear systems are very hard to analyze but there is a lot a very workable theory about linear systems. All elements in a physical system are non-linear if they are over-driven, but the saving grace is that there is a part of the domain of the function that models them where the system is linear. It is in this part of the domain that the elements are used.

Another example of a physical system that is used in its linear part is the audio amplifier. If x represents the input to the amplifier and y represents the output, then the graph of the function that models the amplifier looks something like



The idea of an amplifier is to take a small input function and make it into a big output function where the shape of the graph of the output function is exactly the same as the shape of the graph of the input function, only bigger. In the example, the output is ten times bigger than the input. If the shapes are the same, then the amplifier has good fidelity. As long as the input is between 0 and a, the amplifier is linear and the fidelity is good, but when the input exceeds a, there is distortion. If I like my music loud and increase the input too much, I push the amplifier into the non-linear range and get distortion.

Guitar amplifiers are designed to be linear over a wide range of inputs and this was a problem for early rock bands who wanted distortion. Since they also wanted to be loud, they solved their problem by over driving the amplifiers and in the process fried a lot of amplifiers. Now guitar amplifiers have knobs that give distortion.

With the advent of the modern computer, non-linear problems have become more tractable and the linearity of functions is not the necessity it once was. Still, restricting physical elements to their linear range and using linear functions to model them is probably the most common way to model.

LECTURE 6-10

The decision was made and I am at the base of a mountain. The mango grove and stream seem to be like those at the mountain I first came to, but I am not sure...

I am going to delve into some Ideal World lore that has to do with a class of functions whose graphs are lines. These functions model many things in the Real World but I am going to look at them in a purely Ideal World way.

When physical processes are modeled, the model often falls into a particular Ideal World category of models. In the Ideal World, models in the same category are analyzed in generally the same way, regardless of the physics from which the model arose.

The function f whose rule is f(x) = 10x on the interval [0,10] could model a weighted spring, the position of an object in uniform motion, or the output of an audio amplifier, among many others. The function, f, falls into a category of functions.

The functions in this category all have rules of the form

$$y = f(x) = kx \quad \text{where } k \text{ is a fixed number}$$

and their domains consist of all the real numbers. These are the linear functions where f(0)=0.

The functions in this category of linear functions have two properties that are not shared by other linear functions.

1. $f(x + y) = f(x) + f(y)$.

In fact, $f(x+y) = k(x+y) = kx + ky = f(x) + f(y)$.

2. $f(cx) = cf(x)$.

This is true because $f(cx) = k(cx) = c(kx) = cf(x)$.

It is interesting to note the ambiguity in the use of parentheses in this last statement. The expression ‘f(cx)’ represents the rule of the function evaluated at the number c times x. The expression ‘k(cx)’ represents the number ‘k’ multiplied times the number ‘cx’. Context is everything. If I saw the expression ‘f(cx) = k(cx) = c(kx) = cf(x)’ as graffiti on a wall, I would probably interpret the ‘f’ as a function name and the ‘k’ as a number but the writer may have had something else in mind. Who knows?

If f is a linear function, $f(x) = mx + b$, where b is not zero, then f satisfies neither property. In this case,

$$f(x+y) = m(x+y) + b = mx + my + b$$

and

$$f(x) + f(y) = mx + b + my + b = mx + my + 2b.$$

$f(x) + f(y)$ is not equal to $f(x + y)$.

A similar thing happens with $f(cx)$ and $cf(x)$. Indeed,

$$f(cx) = m(cx) + b$$

and

$$cf(x) = c(mx + b) = c(mx) + cb = m(cx) + cb,$$

so that unless $c = 1$, $f(cx)$ is not equal to $cf(x)$.

The critical condition is that $f(0) = 0$. A linear function belongs to this particular class of functions if and only if $f(0) = 0$.

I have seen that if f is of the form, $f(x) = kx$, then $f(x + y) = f(x) + f(y)$ and $f(cx) = cf(x)$.

It turns out that if f is a continuous function, the converse is true. If f is continuous and satisfies

1. $f(x + y) = f(x) + f(y)$.
2. $f(cx) = cf(x)$,

then $f(x) = kx$ for some number, k.

At the moment I have no name for this class of functions, which is a little odd because I seem to have a name for almost everything.

There is a fundamental problem in mathematics that arises from this kind of function. It is <u>the</u> problem of linear applied mathematics, which is a huge chunk of applied mathematics.

In this problem, f is a linear function such that $f(0) = 0$. Given a number 'b', find the value of 'x' such that $f(x) = b$. This would correspond to the Real World problem of given the distance an object in uniform motion traveled, find the time it took; or given how much a spring stretched, how much weight was hung from it.

The problem can be stated explicitly as:

Given two numbers ‘k’ and ‘b’ find the value of ‘x’ so that kx = b.

If k is not equal to zero, there is exactly one solution, x = b/k.

If k = 0, then there are two possibilities,

1. If b is not zero, there are no solutions. Any number times zero is zero so kx = 0x could never equal a non-zero ‘b’.

2. If b = 0, then every number is a solution because any number times zero is zero. 0x = 0 for all numbers x.

The three possibilities are:

1. Exactly one solution if $k \neq 0$,
2. No solution if k = 0 and $b \neq 0$,
3. Infinitely many solutions if k=0 and b=0.

I’ll put this into Real World terms. If ‘k’ is speed, ‘x’ is time, and ‘b’ is distance, the problem is to find the time it took an object in uniform motion to travel a distance b if its speed was k. If the speed isn’t zero, there is exactly one time it takes to travel the distance b, x = b/k. If the speed is zero, then the object isn’t moving and there is no value of time that will get the object to move a non-zero distance. If the speed is zero, then it will move a zero distance for all values of time.

It may seem that I am making a big fuss over a pretty ordinary function. After all, the equation, 2x = 4, is probably the second equation that is solved in Algebra I, the first being
2 + x = 4. None-the-less, it is the big equation of applied mathematics. The symbols, k, x, and b don’t represent numbers anymore, they represent functions, but the equation is the same and there are the same three possibilities for solution.

The two properties that functions in this class must satisfy have to do with how the function relates to the two basic arithmetic operations, addition and multiplication. It is a typical question to ask how a function handles a+b and ab. Sometimes they don’t do it very well, but at other times, as in the case of f(x) = kx, they do it very well indeed.

It is also true that addition and multiplication are the only operations needed to define the two properties. This means that if a function were defined on a set where only addition and multiplication were defined, for example the integers, I could still consider the class of functions that satisfies the two properties. I will keep my eye out.

Further Consideration of Chapter 6

I want to indicate how it follows that if a continuous function satisfies

1. $f(x + y) = f(x) + f(y)$
2. $f(cx) = cf(x)$,

then $f(x) = kx$ for some number k.

I first show that $f(0) = 0$. If I let $x = y = 0$, then $f(0) = f(0 + 0) = f(0) + f(0)$, and $f(0) = 0$.

If I let $x = y = 1$, then $f(2) = f(1) + f(1) = 2f(1)$. I think I see the pattern.

$$f(3) = f(2 + 1) = f(2) + f(1) = 2f(1) + f(1) = 3f(1).$$
$$f(4) = f(3 + 1) = f(3) + f(1) = 3f(1) + f(1) = 4f(1).$$

Evidently, $f(n) = nf(1)$ if n is a non-negative integer.

Next I'll go after the negative integers.
To that end, $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$, so

$$f(-x) = -f(x),$$

and in particular,

$$f(-n) = -f(n).$$

Fractions are next to fall under my sword.

$$f(1) = f(1/2 + 1/2) = 2f(1/2) \text{ and } f(1/2) = 1/2\ f(1).$$

$$f(1) = f(1/3 + 1/3 + 1/3) = 3f(1/3) \text{ and } f(1/3) = 1/3\ f(1).$$

I need whip this dead horse no further.

$$f(1/n) = 1/n\ f(1).$$

Combining these results, I get $f(n/m) = n/m\ f(1)$, for all rational numbers, n/m. If I let $k = f(1)$, I have $f(x) = kx$ for all rational numbers, x. Since f is continuous,

$$f(x) = kx \text{ for } \underline{\text{all}} \text{ numbers.}$$

This is the tried and true dodge of transferring behavior on the rationals to behavior on all the numbers.