CHAPTER 5

How do I see thee? Let me count the ways...

LECTURE 5-1

For years I worked at Tom McAn's selling children's shoes, although I had no children of my own. I was a very sharp dresser and people said that I was a sight for sore eyes...

If I am present at the occasion of motion, my eyes give me a picture of the motion. I can see that the object is at different places at different times. If I am watching a car drive down a street, I can see it pass two successive light poles at two different times. I can see the distance between the two light poles and I can see time pass; a pile of sand growing larger, hands moving around the face of a clock, the apparent movement of the sun across the sky. But after the motion is over, I would like to have some way to recapture the experience.

If the motion is uniform, speed gives me some sense of what it looked like, and if the motion is non-uniform I have average speed.

"I set the cruise control at 65 and settled back."
"Kyle Petty won the Darlington 500 with an average speed of 137.78 mph."

These are both statements in which numbers give me some idea of what the motion looked liked but I want more. I want a picture of the experience. I will begin to satisfy this want by finding a picture that expresses the size or magnitude of a number and use that idea to get a picture of a function. The picture of a function should give some visual experience of the process modeled by the function. As I can't 'see' into the Ideal World, this is a Real World problem.

The symbols used for numbers are pictures that give little, if any, size information. The symbols '2' and 'II' represent a certain number. In terms of counting, 'II' doesn't do such a bad job but does less well giving a picture of extent or magnitude. As far as I can tell, the numeral '2' is just a symbol and I had to memorize what it meant because I could find no clue in the symbol itself. I do not find a lot of size intuition in the symbols used for numbers

Since I am going to be seeing this picture, it seems reasonable to consider what kind of information I get through my eyes. As far as I can see, the only information my eyes give me is color, light intensity and separation. If I want to put a picture of a number on a piece of paper, I have to use color or separation. I don't see how to use light intensity.

Color is a natural to represent the wavelengths of light and so to represent the numbers that represent wavelength. Color is used to represent the temperature of molten steel and this idea was carried over to the 'color temperature' of film.

On the other hand, people do not perceive colors in the same way, so the color that I use to represent '5' may mean '4' to the person who reads it. Even the same person perceives colors differently if the surrounding colors change, so while it is possible to use colors to represent numbers, the problems outweigh the advantages. I'll see what I can do with separation.

represents a bigger number than

The separation that I am talking about is the separation between the left end of the line segment and the right end.

I could just as well have said

* * represents a bigger number than * *

but I thought that filling in the line between the two points added visual clarity.

I can see that one of the numbers represented is larger than the other, even though I don't know what either of the numbers is. It would be nice to actually get the number itself into the picture, not just relative size. To this end I can make the length of the segment, in some system of units, equal to the number I want to represent. I can chose centimeters for the units and then the number '3' would be represented by a line segment 3 cm. long. This works great for '3' but not so great for '1000'. If I use a standard size paper, I have about 7 inches to put the line segment in, and '1000' would need 10 meters. There are problems on the small end as well. The segment representing a 0.1 would be a millimeter long and smaller numbers would look like indistinguishable dots.

I use this idea but not with a standard unit. I use a 'tailor made' unit that keeps the lengths small enough to fit on the page and big enough to be visible.

If **** represents 1.0 second then ************ represents 3.0 seconds.

If **** represents 10 miles then ************ represents 30.0 miles.

If **** represents 0.01 then ************ represents 0.03.

The lengths that I use to represent numbers are **scaled lengths** and so I will call them until it is bothersome to say 'scaled' and then I will just call them 'lengths'.

If I have a set of numbers which range from very small to very large, say, from 1 to 10,000, then I can't use this technique to represent them. If 1 is represented by a length big enough to be seen, then the length that represents 10,000 won't fit on the page. If the length that represents 10,000 fits on the page, 1 is represented by a tiny dot.

I can not be all things to all sets of numbers. I restrict myself to sets of numbers that can be represented by scaled lengths that can fit on the piece of paper I'm using. This will be true of the numbers that come up in modeling most of Real World physics and consequently will be true of the domain and range of the functions that model physics.

I have a pictorial way to represent numbers but what I am ultimately after is a picture of a function. The very word 'picture' implies Real World. Pictures are Real World things since they exist to be seen, and seeing is a Real World activity. The whole idea of using separation to represent a number, making use of my visual perception of size to see a number, is a Real World idea.

But functions are Ideal World objects and are not in the Real World where I draw. It is like drawing a picture of a dragon; or a picture of saintliness. As a matter of fact, artists approach this problem and they do paint pictures of dragons and angels. They paint pictures that represent abstract ideas. This gives me hope.

I think that there are pictures in the Ideal World too, but they exist only in my mind as concepts. If the number 2 is represented by an Ideal World centimeter, then 6 is represented exactly by an Ideal World line segment exactly 3 centimeters long. I can imagine this but I can't see it.

If the number 2 is represented by a Real World centimeter, 6 is represented by a line segment as close to 3 cm. long as I can measure it. <u>The Real World line segment drawn on the paper is a picture of the Ideal World line segment</u>.

I am going to design a Real World picture of a function and let the Real World motivate the design. The Real World is where I live. That's where it matters. My definition of the Ideal World picture of a function will be modeled after the Real World picture I design.

The design criteria for my Real World picture are that the picture consists of marks made on a Real World piece of paper, that I can in some way 'see' the numbers in the domain and range, and that I can 'see' how the rule associates the numbers in the domain to the numbers in the range.

I don't want the picture to require great artistic skill nor fussiness in measuring. Ballpark measuring is usually good enough for lengths. Say I use five stars ***** to represent the number, 1. The number π is about 22/7, which is a little bigger than 3, so I could represent π by ****************, which if I counted correctly is 16 stars long. There are people who need these pictures drawn with great accuracy, but not me.
These pictures are not unlike abstract art. I look at a painting and there's a guitar and there's a nose and there's an orange. They don't look quite like the guitar, nose and

orange that I’m used to, but I can tell what they are.

So my friend looks at

1   *****
π   *******************

and says, “Well, that π seems a little closer to 4 than 3 to me, but if you say that’s a picture of π, it’s OK with me.”

The drawing ******************* is not the number π, it is a picture of π, and I take full advantage of artistic license.

LECTURE 5-2

One evening as I watched the horizon at moon rise, a blazing ball of fire flashed across the darkening sky. It was the peak experience of my life and seemed to have a significance far beyond its visual impact. "I must put this on canvas", I thought. I left Tom McAn's and began to study art...

If the idea of representing numbers as the distance between two points or the length of a line segment is viable, I should be able to use it to draw a picture of the function that models uniform motion. Hopefully this picture will satisfy my design criteria and I can then use the experience that I gain drawing this picture to make a general definition of the Real World picture of a function.

An object is moving in a straight line with uniform motion and a speed of 2 feet\second. As the object passes the point that I have designated as the zero point, I start the clock at zero and I observe the motion for 20 seconds. I let t be the independent variable, time, and s be the dependent variable, distance. I model the position of the object by the function s whose domain is [0,20] and whose rule given by s = s(t) = 2t. This is the function I want to draw a picture of.

The only Real World numerical knowledge I have about the function is what I get by evaluating the rule at different values of time, so I do that and get the table

t in seconds	s in feet
0	0
1	2
2	4
3	6
10	20
15	30
20	40

I would hope that at least this information would be in the picture. I am going to represent the distance numbers by line segments and I need to choose a scale. If I let one star be two feet, forty feet looks like ********************. I think this is too short. I want to use up more of the page. If I let one star be one foot, then forty feet looks like **. This looks OK to me.

Now the table looks like

```
time
 0
 1    **
 2    ****
 3    ******
10    ********************
15    ******************************
20    ****************************************
```

This is not a bad looking picture but if I am scaling the distance, I should probably scale the time as well. In the picture there is the same distance between 2 seconds and 3 seconds as there is between 3 seconds and 10 seconds, but the intervals of time are quite different. I let one space down represent one second.

```
 0
 1 **
 2 ****
 3 ******






10 ********************




15 ******************************




20 ****************************************
```

Not bad.

I think I'll try it with just the end points of the segments.

```
 0
 1   *
 2     *
 3       *



10                 *



15                          *



20                                  *
```

Even better. I may be looking through rose colored glasses, but I think this picture ‘looks’ uniform. I think I am on the right track.

There are applications where this picture would be just the ticket, but it is not the traditional picture. Since the point of the picture is to make the idea behind the application clearer, the application must ultimately determines the picture I use, not tradition. The traditional way is, however, comfortable for me and more often than not it works just fine.

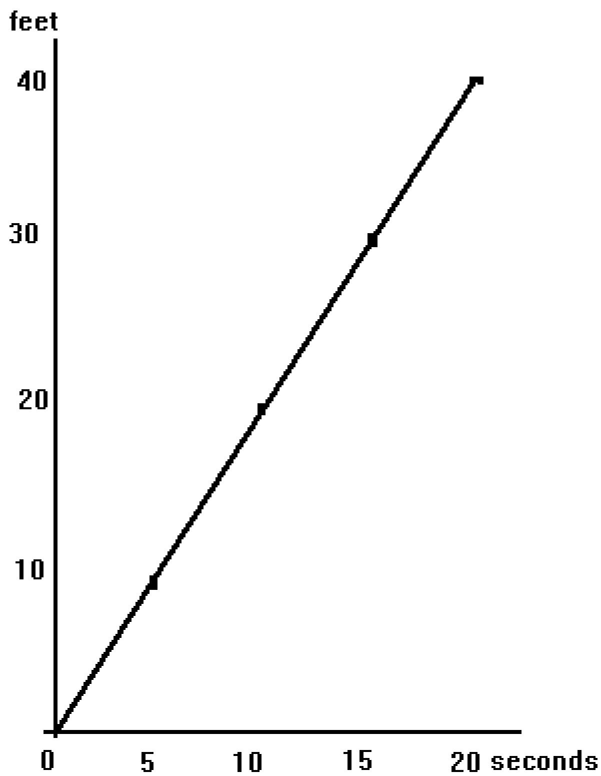
Traditionally, 'up' is positive, 'to the right' is positive and, the independent variable is put on the horizontal. I do it this way unless there is a compelling reason not to. If I put these traditions into effect, I get the following picture:

```
feet
40                                      *



30                             *



20                   *






 6      *
 4    *
 2  *
 0  1 2 3            10        15        20        seconds
```

I'm going to call a pair of numbers that give the time and the distance at that time a **state** of the object. Each point in this picture represents a state of the object in motion. The point above 3 and to the right of 6 represents the state of the object 3 seconds after the clock started.

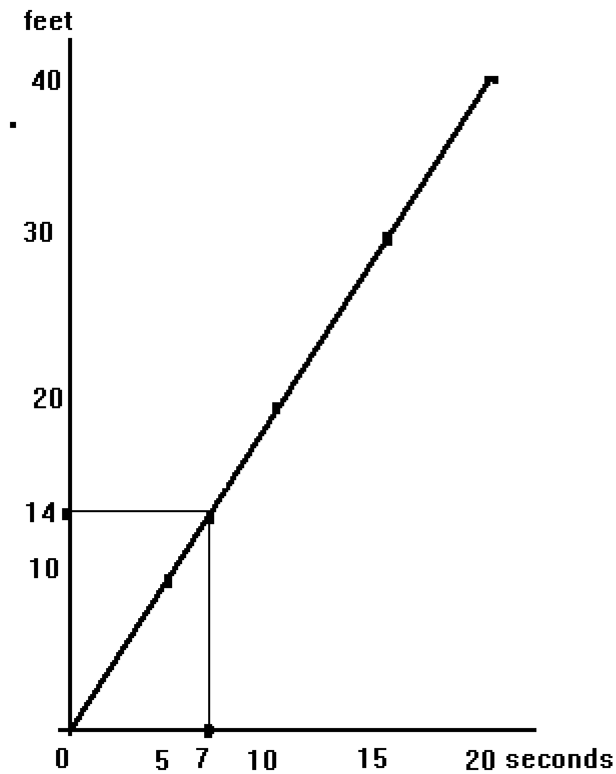
The numbering on the horizontal looks a little odd because of the bunching at low values of time. I think it would look nicer to have the numbers equally spaced but that is just my idea of orderly.

For my final effort I'm going to draw a horizontal line through 0 and put numbers on it that represent time every five seconds. This determines my time scale. I'll draw a vertical line through 0 and put numbers on it that represent distance every ten feet. This determines my distance scale. The horizontal line is called the **horizontal axis** and the vertical line is called the **vertical axis**. I am going to give in to my orderly nature and evaluate the rule every five seconds. Finally, after I have drawn all the points, I am going to 'kick the can' and draw the line that obviously passes through them. I don't really have a reason for drawing the line except for my obsession to 'kick cans'. Here goes.



Now, this is just not too bad. I think I'll go with this as my Real World picture of the Ideal World function that models the uniform motion of the object.

Once I have drawn the line, its significance seems clear. Looking at the picture, I think that it surely must be true that every point on that line is a state of the object. If there is any justice in the universe at all, I should be able to use this picture to find the distance the object has moved at times other than the data points. For example, if I want to find how far the object traveled in seven seconds, I should be able to go up vertically from 7 until I hit the line at some point. I can use the scale on the vertical axis to see how far this point is from the horizontal axis. I want this number to equal the number I get if I apply the rule,
$s(t) = 2t$, to $t = 7$. Here I am, wanting something again.



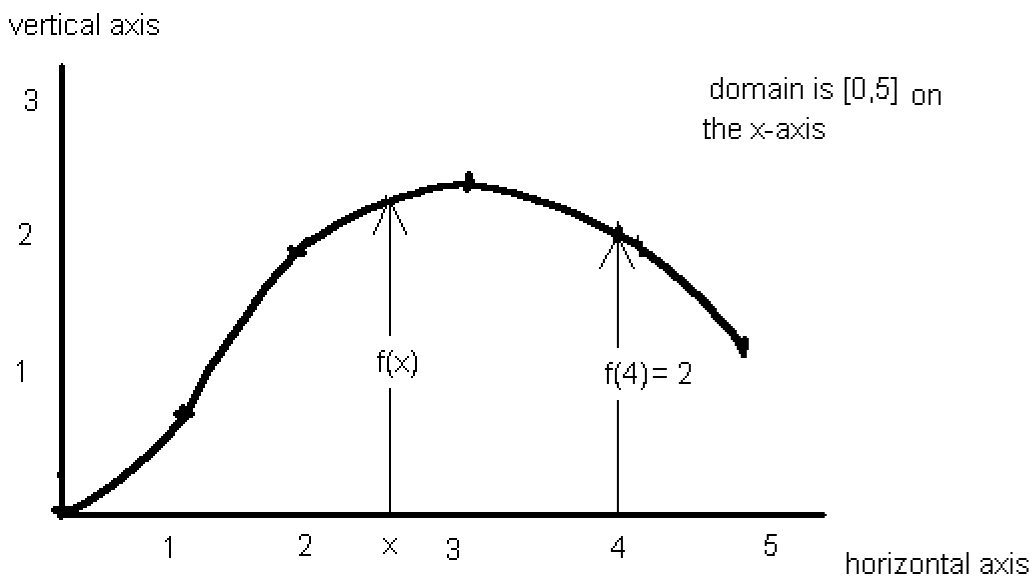
It looks like the vertical line through 7 hits the line at 14 on the vertical scale. The rule gives s(7) = 2 x 7 = 14. I am always pleased to find examples of justice in the universe. I think that every point on that line represents a state of an object in uniform motion with speed 2.

I am swimming in some kind of Real World/Ideal World stew. I draw a Real World picture of the Ideal World function using a finite number of Real World data points and following the dots, which is certainly a Real World activity. Always hoping for the best in the best of all possible worlds, I think that I can use the 'follow the dots curve' to evaluate the Ideal World rule by measuring from the 'follow the dots curve' to the horizontal axis.

Even though an Ideal World number can not be determined exactly by a Real World measurement, the measurement can give a useable approximation and that is as close as I ever get to an Ideal World number anyway. The picture works as well as a calculator to compute numbers if the accuracy required is not too great.

Based on my experience with the function that modeled uniform motion, I am going to design my Real World picture of a function whose rule can be evaluated. If I can't evaluate the rule of a function, I can't draw a Real World picture of it.

I am given a function that has a rule I can evaluate. I draw a horizontal axis and a vertical axis. I mark off scaled distances on the horizontal axis, starting at the intersection of the axes, to represent values of the independent variable, and mark off scaled distances on the vertical axis to represent values of the dependent variable. I then evaluate the rule for some finite number of values, x, and put dots on the paper a scaled distance f(x) from the point x on the horizontal axis. Finally, I will follow the dots with a smooth curve. *Voila`*, my picture. I see the domain as points on the x-axis and the range as points on the y-axis. The curve itself lets me see how the rule associates numbers in the domain with numbers in the range. A Real World picture of a function is called a **graph of the function**.



I am going to end this lecture with the picture of the function that models motion, $v = v(t) = 2$. This is a picture of a line.

```
speed
ft./sec.
 4
 3
 2*******************************************************
 1
 0  1  2  3  4  5  6  7  8  9  10  11  12  13  14  15  16  17  18  19  20    time sec.
```

LECTURE 5-3

I mastered impressionism, surrealism, expressionism, realism, and dada. They were all grist for my mill...

I have a method of drawing a picture of a function in the Real World and I have called this drawing the **graph of the function**. I am reasonably pleased with what I have done but the phrase 'picture of a function' continues to nag at me. Just what am I drawing a picture of? A function is not like a horse, which can be drawn and the drawing compared with the actual horse. A function is an intellectual concept and what I draw and see can at best stir some understanding of the concept. There is nothing visual to compare it with. Courage and chairs are both in the Ideal World and pictures of both are drawn in the Real World, but, it is more straight forward to draw the chair and it is easier to see if the drawing looks like a chair than if it looks like courage. I would like to have some object in the Ideal World that is to the graph of a function as the ideal chair is to its picture in the Real World.

Well, the 'graph of a function' is a Real World thing and I can consider modeling it in the Ideal World. The Real World 'graph of the function' is the picture of its model in the Ideal World. Of course I'm going to call this Ideal World model of the 'graph of the function', the '**graph of the function'** also. I will often preface 'graph' with Real World or Ideal World to keep track of which one I mean.

I have considered modeling the graph in the Ideal World and now I shall actually do it. My Real World graph starts with two perpendicular line segments, the axes, on a piece of paper. Each of these line segments has numbers marked on it. The domain of the function is represented by numbers on the horizontal axis, and the range of the function is represented by numbers on the vertical axis.

I will model the two perpendicular, numbered line segments by two intersecting, perpendicular Ideal World number lines , which I also call **axes**. The piece of paper will be modeled by the plane determined by these two lines. The Ideal World graph of a function will be part of this plane. The axes of my Real World graph are line segments a few inches long while the Ideal World axes are lines from Euclidean geometry and are infinite in extent. The piece of paper fits easily on my desk while the plane determined by the axes is from Euclidean geometry and goes on forever.

<u>I would like to say a few words about number lines.</u> The numbers on the axes of the Real World graph are scaled distances. The number lines are models of the Real World process of scaling and I want to talk about how the numbers are <u>identified</u> with the points on the line.

I can make any line into a number line so I'll take <u>any</u> line and the task is to assign all the points on the line to numbers. I choose <u>any</u> point on the line as the point I assign to the number zero. This is such an important point that I give it a name. I call it the **origin**.

Next I choose any other point on the line and assign this point to the number 1. <u>The separation between these two points is what I take for the unit distance and measure all distances on the line with respect to it</u> . This is the Ideal World model of Real World scaled distances. There is another point on the line whose distance from the origin is 1 and I assign this point to the number, -1. The origin divides the line into two sides; one side contains the point assigned to 1 and the other side contains the point assigned to -1.

Assigning a point to the number 1 does two things. First, it picks the side of the line that has the points I assign to the positive numbers and so also, it picks the side whose points I assign to the negative numbers. I'll call the side that has the point assigned to 1, the **positive side** and the other side, the **negative side**. Second, it determines the unit distance I'm going to use.

When I began by dropping a rock, so long ago, distance was a non-negative number that measured the separation of two points in space. This number was determined using some sort of measuring stick and the number varied depending on what unit distance I used. The separation of two points could be described by the numbers 1 foot, 12 inches, 30.48 centimeters, or 0.0001893 miles, among many others. If I were going to measure distance on the line using one of these units, my point of view would be that there is some pre-existing way to measure distance and I assign points on the line to numbers with respect to this distance. If I am using centimeters, I assign a point 1 cm. from the origin to the number, 1. The separation between two points doesn't change but the numbers that represent the separation change when I change units and where I put 1 on the line.

Usually standard units don't work very well and I have to scale the distances. To do this I pick both a point I assign to zero and a point I assign to 1 anywhere on the line, and call the separation between these two points the unit distance. The separation between <u>any</u> two points on the line is now measured in terms of this unit of length. While I can chose <u>any</u> two points, I try to chose two that make life easier for me. This is a 'tailor made' unit of length and I, personally, always use it.

Anyway, once I tie down the unit distance, I assign each point on the positive side of the line to the positive number that gives the distance of the point from the origin. I assign each point on the negative side of the line to the <u>negative</u> of the positive number that gives the distance from the point and the origin.

If I have any positive number, there is exactly one point that far from the origin on the positive side of the line and exactly one point that far from the origin on the negative side. The origin is assigned to the number 0. This means that <u>every</u> number is assigned exactly one point on the line.

Every point on the line is exactly one distance from the origin and is assigned to exactly one number, which is positive or negative depending on which side of the origin the point lies.

When all is said and done, every number is assigned exactly one point on the line and every point on the line is assigned to exactly one number. There is a one-to-one correspondence between <u>all</u> the numbers and <u>all</u> the points on the line.

The unique number assigned to a point on the line is called the **coordinate** of the point.

I have assigned all the numbers to all the points on a line but I am going to consider the association between point and number as more than just an assignment. I am going to <u>identify</u> the point and the number.

I will talk about 'the point, 2, on the line' but 2 isn't a point, it's a number, and numbers aren't on the line, points are. And a point isn't a number, it's a point. I talk about the 'point 2' because in my mind I have <u>completely identified</u> the point and the number. When I say, 'the point 2', I mean the point that has been assigned to the number 2. When I think about the number, 2, <u>in this context</u>, I'm thinking about the point on the line that is assigned to the number, 2. The origin is the point, 0. The numbers are no longer all jumbled up in a huge basket, they are neatly arranged along the line.

So I model the axes of the Real World graph with two perpendicular number lines that meet at their origins. These number lines are called **coordinate axes**. The two coordinate axes together are called a **coordinate system.** The plane determined by the two coordinate axes is called the **coordinate plane** or the **Cartesian plane** after Rene Descartes who originally had the idea.

The axes in the Real World graph are called the horizontal axis and the vertical axis. This is a problem in the Ideal World because there is no horizon and no obvious choice for the horizontal axis. Since I put the numbers from the domain of a function on the horizontal axis of the Real World graph, I need a way to distinguish a horizontal axis in the model.

The problem here is an Ideal World / Real World problem. Horizon is a concept connected with sight, which the Ideal World doesn't understand. <u>The Real World axes are what they are because of how they look</u> so I want to be able to see the axes in the Ideal World; I want to 'see' that they 'look' right. But where? There is no 'place' in the Ideal World to see them. There is no latitude and longitude. Unbidden comes a picture to my mind of crossed axes hovering over rolling green hills, a farm house and barn in the distance. I flee with my axes to outer space where there are no reference frames to distract me. I float around this object and there is <u>nothing</u> to see except the axes.

I make the choice of horizontal by *fiat*. I point to one of them and say, "You, you are the horizontal axis." I turn to the other and say, "And, you, you poor bastard, are the vertical." I position myself so that if the positive side of the horizontal axis is on my right, the positive side of the vertical axis is up. What I 'see' in my imagination looks like the axes that I draw on my paper.

I believe, now, that there is an object in the Ideal World which if I could see it, would look like the axes of a Real World graph.

My imaginary Ideal World is <u>not</u> the Ideal World and so my discussion lacks rigor. But, in order to talk about 'my right' and 'up', I have to somehow be there, and so this is how I think about it. I slack off a little on the rigor now and then for the sake of understanding.

I now draw a Real World picture of the Ideal World axes. My paper defines horizontal and I use a ruler to draw a line across the middle of the paper. This is my rendition of the horizontal axis. I draw a vertical line down the middle of the paper for the vertical axis. I place a 1 on both axes and place other numbers on the axes with respect to these unit lengths. My drawing is complete. Probably not the best thing I ever drew, but it has its charm, not the least of which is that my drawing looks like the axes I drew for the Real World graph.

LECTURE 5-3

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I have considered modeling the graph in the Ideal World and now I shall actually do it. My Real World graph starts with two perpendicular line segments, the axes, on a piece of paper. Each of these line segments has numbers marked on it. The domain of the function is represented by numbers on the horizontal axis, and the range of the function is represented by numbers on the vertical axis.

I will model the two perpendicular, numbered line segments by two intersecting, perpendicular Ideal World number lines , which I also call **axes**. The piece of paper will be modeled by the plane determined by these two lines. The Ideal World graph of a function will be part of this plane. The axes of my Real World graph are line segments a few inches long while the Ideal World axes are lines from Euclidean geometry and are infinite in extent. The piece of paper fits easily on my desk while the plane determined by the axes is from Euclidean geometry and goes on forever.

<u>I would like to say a few words about number lines.</u> The numbers on the axes of the Real World graph are scaled distances. The number lines are models of the Real World process of scaling and I want to talk about how the numbers are <u>identified</u> with the points on the line.

I can make any line into a number line so I’ll take <u>any</u> line and the task is to assign all the points on the line to numbers. I choose <u>any</u> point on the line as the point I assign to the number zero. This is such an important point that I give it a name. I call it the **origin**.

Next I choose any other point on the line and assign this point to the number 1. The separation between these two points is what I take for the unit distance and measure all distances on the line with respect to it . This is the Ideal World model of Real World scaled distances. There is another point on the line whose distance from the origin is 1 and I assign this point to the number, -1. The origin divides the line into two sides; one side contains the point assigned to 1 and the other side contains the point assigned to -1.

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I have assigned all the numbers to all the points on a line but I am going to consider the association between point and number as more than just an assignment. I am going to identify the point and the number.

I will talk about 'the point, 2, on the line' but 2 isn't a point, it's a number, and numbers aren't on the line, points are. And a point isn't a number, it's a point. I talk about the 'point 2' because in my mind I have completely identified the point and the number. When I say, 'the point 2', I mean the point that has been assigned to the number 2. When I think about the number, 2, in this context, I'm thinking about the point on the line that is assigned to the number, 2. The origin is the point, 0. The numbers are no longer all jumbled up in a huge basket, they are neatly arranged along the line.

So I model the axes of the Real World graph with two perpendicular number lines that meet at their origins. These number lines are called **coordinate axes**. The two coordinate axes together are called a **coordinate system.** The plane determined by the two coordinate axes is called the **coordinate plane** or the **Cartesian plane** after Rene Descartes who originally had the idea.

The axes in the Real World graph are called the horizontal axis and the vertical axis. This is a problem in the Ideal World because there is no horizon and no obvious choice for the horizontal axis. Since I put the numbers from the domain of a function on the horizontal axis of the Real World graph, I need a way to distinguish a horizontal axis in the model.

The problem here is an Ideal World / Real World problem. Horizon is a concept connected with sight, which the Ideal World doesn't understand. The Real World axes are what they are because of how they look so I want to be able to see the axes in the Ideal World; I want to 'see' that they 'look' right. But where? There is no 'place' in the Ideal World to see them. There is no latitude and longitude. Unbidden comes a picture to my mind of crossed axes hovering over rolling green hills, a farm house and barn in the distance. I flee with my axes to outer space where there are no reference frames to distract me. I float around this object and there is nothing to see except the axes.

I make the choice of horizontal by *fiat*. I point to one of them and say, "You, you are the horizontal axis." I turn to the other and say, "And, you, you poor bastard, are the vertical." I position myself so that if the positive side of the horizontal axis is on my right, the positive side of the vertical axis is up. What I 'see' in my imagination looks like the axes that I draw on my paper.

I believe, now, that there is an object in the Ideal World which if I could see it, would look like the axes of a Real World graph.

My imaginary Ideal World is <u>not</u> the Ideal World and so my discussion lacks rigor. But, in order to talk about ‘my right’ and ‘up’, I have to somehow be there, and so this is how I think about it. I slack off a little on the rigor now and then for the sake of understanding.

I now draw a Real World picture of the Ideal World axes. My paper defines horizontal and I use a ruler to draw a line across the middle of the paper. This is my rendition of the horizontal axis. I draw a vertical line down the middle of the paper for the vertical axis. I place a 1 on both axes and place other numbers on the axes with respect to these unit lengths. My drawing is complete. Probably not the best thing I ever drew, but it has its charm, not the least of which is that my drawing looks like the axes I drew for the Real World graph.

LECTURE 5-4

But no matter what I tried, I could never quite recapture the experience of the fireball...

There are a few odds and ends I want to pick up before I go on.

I already have a way to talk about certain collections of numbers, in particular the intervals, and now these sets of numbers are identified with sets of points on the line.

[2,5] = { x | 2≤ x ≤5 }denotes the set of numbers between 2 and 5 inclusive. These are exactly the numbers that are identified with the points on a number line that lie on the line segment between the point, 2, and the point, 5. The interval notation is a convenient way to represent segments on a number line. To me, it emphasizes the identification of the points and numbers and I find this a ‘plus’. I find anything that keeps my mind on what’s going on a ‘plus’.

Not only does [2,5] denote the set of numbers between 2 and 5 inclusive, it also denotes the set of points on the positive side of a line whose signed distances from zero are between 2 and 5 inclusive.

There are four variations on this notational theme depending on what I want to do with the end points of the interval, which is now also a line segment.

```
[a , b] = { x | a = ≤ x ≤b} --------[------------]------------
                                     a            b
[a , b) = { x | a ≤ x < b} ----------[------------)-------------
                                     a            b
(a , b] = { x | a < x ≤ b}----------(-------------]--------------
                                     a            b
(a , b) = { x | a < x < b}-----------(-------------)---------------
                                     a            b
```

Another odd end is that I am running afoul of the awkwardness of negative numbers and in all candor I must admit that I have a little prejudice against negative numbers. It’s not that I wouldn’t lend one money but I don’t seek their company. In 1831 De Morgan felt that
(0 - 2) was as meaningless as $\sqrt{-2}$. Euler, in the eighteenth century, thought that the negative numbers were bigger than infinity. I, also, have a conceptual problem with something that is less than nothing.

I view a negative number as representing the opposite of what a positive number represents. If I consider work into a system as positive, then work out of the system is negative. This is how I think of it if I am putting work into the system, say lifting a sack of cement. If I am the recipient of the work, I might consider the work out of the system positive and the work into the system negative. Either way works OK as long as I am consistent.

If the time after the clock starts is positive, then time before the clock starts is negative. I think of the money I owe the bank as negative and the money the bank owes me as positive. The bank looks at this the other way around. I do not look at my debt as some 'less than nothing' number, I look at it as opposite to my assets.

Distance is an inherently positive quantity and I can't travel a distance that is less than zero. On the other hand, I can travel forward and backward and I would like a way to distinguish between these opposites. Since 'opposites' are grist for the mill of the negative number, it would seem that the minus sign could be used to distinguish movement that was opposite to movement that I call positive.

Now, when I look at a number line, being a point 3 units on one side of the origin is the opposite of being a point 3 units on the other side of the origin. The minus sign seems appropriate. A minus sign in front of a number moves the number to its opposite number, so to speak, on the other side of zero. The number -3 is on the opposite side of zero from 3. The number -(-3) is on the opposite side of zero from -3, so -(-3) = 3. The use of the minus for the coordinates of points on the side of the line that is opposite to the positive side of the line is consistent with my feeling that minus means 'opposite'.

The rather formal flip-flop of a point upon the application of a minus sign works in an operational way, but I want to keep distance in the picture. If P is some point on the line, I want an easy way to <u>say</u> what its coordinate is. Since there are two cases, depending on which side of 0 the point is, it is awkward to give the coordinate in terms of distance. If a point is on the positive side of the line, the coordinate is the distance of point from the origin. If a point is on the negative side of the line, then its coordinate is the negative of the distance of the point from 0.

I am going to use an *ad hoc* quantity that I call '**signed distance**' that will accommodate the two cases by incorporating the minus sign into its definition. Signed distance to the left of a point is a negative number, signed distance to the right of a point is a positive number. The point, 2, is a signed distance of -3 from the point, 5; the point, 5, is a signed distance of 3 from the point, 2. The point, 4, is a signed distance of 4 from the origin, the point, -4, is a signed distance of -4 from the point,0.

The coordinate of a point on the line is the signed distance of the point from the origin. That is pretty concise.

My model of the two Real World axes is two perpendicular coordinate axes which determine a coordinate plane. The points on the horizontal axis are going to represent numbers that the independent variable of a function can take. If I am using x for the independent variable, I call this axis the x-axis. If I use t for the independent variable, I call it the t-axis. The points on the vertical axis are going to represent numbers that the dependent variable takes. If I use y for the dependent variable, I call it, not surprisingly, the y-axis. I know of no law of God or man that says the independent variable must go on the horizontal axis but it always does; an inflexible tradition.

The coordinate plane is of infinite extent and the coordinate axes go on forever and I cannot draw this on a piece of paper. Fortunately there is usually only a small part of this plane that I am interested in and I try to get that represented on the paper.

I have said that distance was a non-negative number as opposed to a positive number. The distance between two points is inherently positive except if the two points are the same. If I say, " I have two points.", I must decide if I mean that there are really two distinct points or if I am willing to allow the possibility that the two points are the same. I choose the later for convenience. I have been considering an object's motion where I want distance to be zero when time is zero. If I did not make this choice, I would have to consider $t=0$ and $t > 0$ as separate cases.

The only time that the distance between two points is zero is when the points are the same.

No matter how much of the axes and plane I draw on the paper, it is such a tiny part of the Ideal World axes and plane. Once you allow infinity, everything else is zero. If I take the area of the part of the plane that I have on my paper and throw it away, I have not diminished the area of the plane. Things like that happen in the Ideal World. I keep coming back to the ideas of big and small.

In the Real World, things are big or small compared to me. There are things that are bigger than I am; buildings, mountains, the moon, the sun, the solar system, the galaxy, and so on. These are things I call big. There are things smaller than I am; dogs, cats, mice, microbes, and a virus. I call these small.

There is no one except God in the Ideal World, so there is no 'big' or 'small' in the way that I think of big and small. I can't deal with things that I think are very large or very small because of my size. I can't deal with a virus because it is too small and the universe is too big for me to comprehend. In the Ideal World, God does not have a personal size to prejudice the judgment as to big or small and so does not have this problem. The interval $[0, 10^{-45}]$ is not much different than the interval $[0, 10^{45}]$ in the eyes of God. One of them is quite a bit longer than the other but they are both positive length intervals and they both have the same number of points. But I do have a personal size and when the time comes to draw on paper, I will have to have to accommodate this bias.

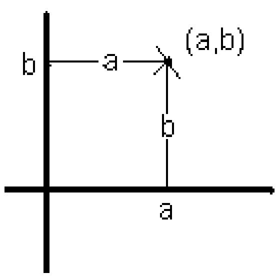
LECTURE 5-5

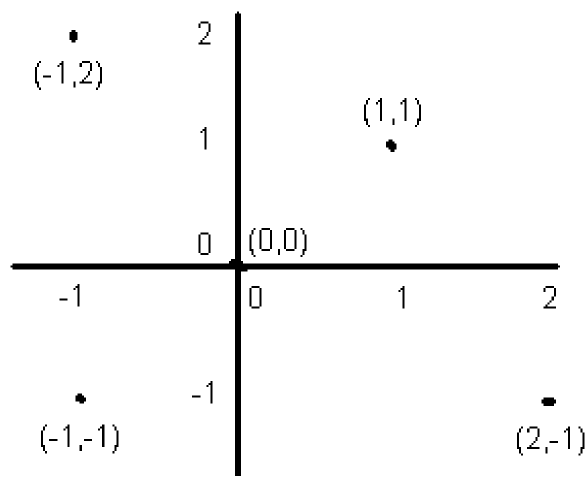
I decided that I needed to consider the deeper significance of the fireball. Perhaps the answer was in philosophy...

Every point on the pair of axes of a coordinate system is identified with a number. I now identify every point in the coordinate plane with a pair of numbers.

Every point in the plane is a signed distance from each of the two coordinate axes and I identify these two numbers with the point. I should remark that by ‘the distance of a point from a line’ I mean the perpendicular distance.

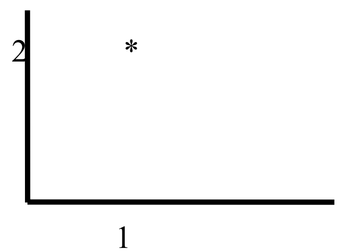
The point that is a signed distance ‘a’ from the vertical axis and a signed distance ‘b’ from the horizontal axis is <u>identified</u> with the **ordered pair**, (a,b). The numbers ‘a’ and ‘b’ are called the **coordinates** of the point and every point has a unique pair of coordinates. On the other hand, if (a,b) is any ordered pair of numbers, there is a point that is a signed distance, a, from the vertical axis and a signed distance, b, from the horizontal axis and this point must have coordinates, (a,b). <u>There is a 1 to 1 identification between all the points in the coordinate plane and all the ordered pairs of real numbers</u>.

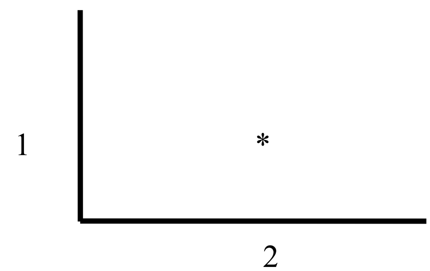




If I have called the horizontal axis, the x-axis, and the vertical axis, the y-axis, then I would call ‘a’ the x-coordinate and ‘b’ the y-coordinate. The two axes meet at their respective origins and this meeting point has coordinates (0,0). I call this point the **origin** of the coordinate system.

The point that is a signed distance 1 from the vertical axis and a signed distance 2 from the horizontal axis is identified with the ordered pair, (1,2).

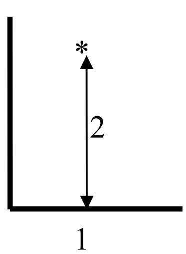


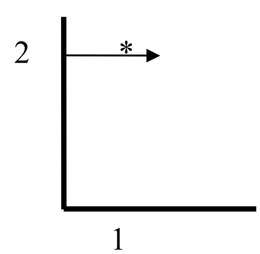


The point that is a distance 1 from the vertical axis and a distance 2 from the horizontal axis is not the same as the point that is a distance 1 from the horizontal and a distance 2 from the vertical axis. This means that the order in which the numbers are given does make a difference and I must use ordered pairs. I have chosen the convention that the first number given is the signed distance from the vertical axis and the second is the signed distance from the horizontal axis.

Points on the coordinate axes wear two hats, one as a point on a line and the other as a point in a plane. I can talk about the point 1 on the horizontal axis or I can talk about the point (1,0) in the coordinate plane. The point 2 on the vertical axis is the same as the point (0,2) in the coordinate plane.

There are different ways to view the position of the point (a,b) in the coordinate plane. I can think of (1,2) as being 2 units directly above the point 1 on the horizontal axis,





or one unit directly to the right of the point 2 on the vertical axis. I try to think of it both ways but context can emphasize one view over the other. I tend to think of the vertical axis as a ruler to see how far the point is from the horizontal axis, so my natural inclination is to see (1,2) as being 2 units directly above the point 1 on the horizontal axis.

It is customary to use the symbol x to represent the points on the horizontal axis and I follow this custom unless there is a compelling reason to use another symbol like t, if the numbers on the horizontal axis represent time. I cannot help but notice that I have talked about points and numbers on the horizontal axis in the same breath. I can use them interchangeably because I have identified the points with the numbers.

Now that I have given coordinates to all the points of the coordinate plane, I am prepared to say what I mean by the **graph of a function** in the Ideal World. If f is a function whose rule is expressed y = f(x), then the **graph of *f*** is the set of points in the plane

$$\{ (x , f(x)) \}$$ where x assumes all the values in the domain of f.

I think of the number x as being identified with a point on the x-axis so I think of the rule as being evaluated at the point, x, on the x-axis. I think of going up or down from the point, x, on the x-axis, a signed distance, f(x), and there find the point (x,f(x)) on the graph of f . I 'think' because this is all in the Ideal World and I can't see or draw anything. As soon as I start to draw, I'm dealing with the Real World graph.

Every point on the graph of f is of the form (x,f(x)) where x is in the domain of f. This means that if (a,b) is a point on the graph of f , then, necessarily, 'a' is in the domain of f and b = f(a).

Just to keep what's what in mind,

A function is a rule and a set of numbers.
The Ideal World graph of a function is a set of points in a plane.
The Real World graph of a function is a drawing on a piece of paper.

I have been looking at an object moving at 2 ft./sec. in uniform motion for twenty seconds. The position of the object is modeled by the function s whose domain is the interval of time [0,20], and whose rule is given by s = s(t) = 2t. The graph of this function is a set of points in a coordinate plane where the horizontal axis is the t-axis and the vertical axis is the s-axis.

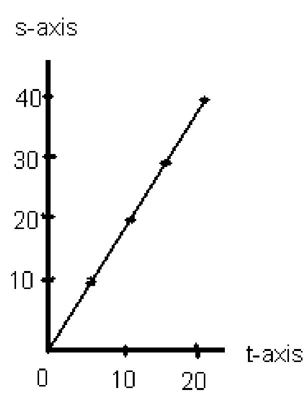
The graph of *s* is the set of points in the plane

{ (t , *s*(t) }= { (t , 2t) } where the rule of *s* is evaluated at all the values, t, in [0,20], that is, at all points, t, in the interval [0,20].

God does not have to follow the dots in the Ideal World because all the dots are there, but unfortunately, only God can see the Ideal World graph. If I want something that makes an impression on <u>my</u> retinas, I must draw a picture of the Ideal World graph on a Real World coordinate plane. To do this I evaluate the rule at a few points, say

t	*s*(t)	(t, *s*(t))
0.0	*s*(0) = 0	(0, *s*(0)) = (0, 0)
5.0	*s*(0.5) = 10	(5.0, *s*(5)) = (5, 10)
10.0	*s*(1.0) = 20	(10, *s*(10)) = (10, 20)
15.0	*s*(1.5) = 30	(15, *s*(15)) = (15, 30)
20.0	*s*(2.0) = 40	(20, *s*(20)) = (20, 40)

and plot the points. They appear to all lie on the edge of a ruler so I'll draw a line through them.



This is the Real World graph of the function and it is a picture of the Ideal World graph of the function..

The coordinate plane is a Euclidean plane of infinite extent and the coordinate axes are straight lines that go on forever. The standard version of the universe is finite and its lines are not what is ordinarily considered straight, whatever straight means. The coordinate plane lives only in the Ideal World.

On the other hand, the universe is considered to be locally flat. This means that in my immediate vicinity, I can measure no difference between the actual universe, whatever it may be, and a Euclidean universe. I have no problem thinking that my piece of paper is part of a Euclidean plane.

If I am standing beside a long, straight road across one of the great, flat expanses of Australia, the road looks to me like a line going on forever across a flat plane. I would model the road as a line on a plane. As I rise, I see that in reality, the road is an arc of a circle on a sphere, and from that vantage point I would model the road in that way.

LECTURE 5-6

Platonism seemed to offer hope and existentialism excited me. Perhaps the fireball was part of Casteneda's Separate Reality...

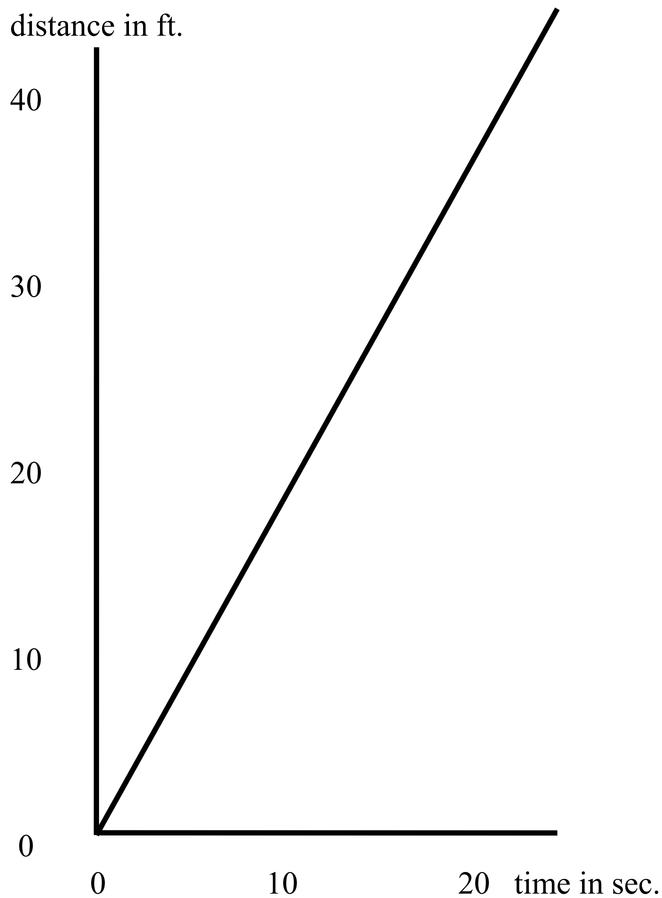
Regardless of the general structure of the universe, my part of the universe is an 8 1/2`` by 11`` piece of paper. I am going to draw the graph of the function s = *s*(t) again and take some time to talk about it.

I will use a ruler to draw a horizontal line and a vertical line for pictures of the axes, but I have to decide where to put them on the paper.

There are two design objectives. One is to have the domain of the function visually accessible on the t-axis and the range likewise on the s-axis. The other is that the picture reflects my intuitive feeling about the nature of the process that the function is modeling, or if the function is not modeling anything, the picture should reflect the nature of the function as a being in its own right. I often can not attain both objectives and must make compromises.

If I want to emphasize the intuitive feeling of the function, I should use the same scale on both axes, or as close to that as I can. If one of the domain or range has very large numbers and the other has very small numbers, then using the same scale on both axes is not possible; if I can see the smaller, I can't get the larger on the paper and I have to re-scale the axes. If I have to re-scale, my Real World picture of the Ideal World graph may be quite distorted, more like surrealism than realism.

This is the graph of the function $s(t) = 2t$ on the interval [0,20].



I have called this picture 'the graph of the function' and did not qualify it with 'Real' or 'Ideal'. Since I can see it, it must be the Real World graph and a modifier isn't needed.

I think this picture gives a good feeling of the function. The rule of the function is $s(t) = 2t$, so that the object moves two feet in every second. The number of feet is increasing twice as fast as the number of seconds and the line appears to be rising twice as fast as it is moving to the right. The steepness of the line seems to give some measure of the speed.

The speed of the object is 2 feet/second and the line rises twice as fast as it moves to the right. I might conjecture that if the speed were 3 feet/second, the line might rise three times as fast as it moves to the right.

I drew this picture using the usual kind of technique for drawing Real World graphs. I evaluated the rule at one second intervals from zero to twenty and put dots on the paper to represent the points on the graph that corresponded to these times. I put a ruler on these points and they all seemed to fall along its edge, so I drew the line. It looks like what is commonly called a line so I call it a line.

The road in Australia also looked like a line but was actually an arc on a very big circle and I suppose the same thing could be happening here. It is possible, I suppose, that if I plotted points every second for a thousand seconds on a very big piece of paper, I would see that it was not a line, but an arc of some curve.

Well, I didn't use a big piece of paper and a thousand points, and what I have looks like a line so I will call it a line. I do, however, see that I am going to have to deal with this problem eventually and develop some way to tell if what I have is really a line or not. This means that I will need a definition of a line so that I can compare what I have with the definition and see if it fits.

I have used positive integers as reference points on the axes, which are both Real World and Ideal World numbers. I can also use strictly Ideal World numbers like π and $\sqrt{2}$ as points on the axes. I am drawing a picture of an Ideal World graph of a function, and it is quite possible that π and $\sqrt{2}$ are noteworthy in this graph and should be included somehow in the Real World picture.

There are Real World numbers that approximate π and $\sqrt{2}$ well enough to position them on the axes. None of these marks on the axes are exact, but this is a <u>picture</u> of exact things, not the exact things themselves. Sixteenth century artists painted halos about the heads of saints. The halo was their 'world of the flesh' way to represent a 'world of the spirit' property of the person they were painting. I use the Real World 1.41 to represent the Ideal World $\sqrt{2}$.

I have been talking about 'the' graph of a function and in the Ideal World, this is correct. A function has only one graph in the Ideal World, a certain set of points in a coordinate plane. In the Real World, it is another matter. The Real World graph of a function is a picture and I can draw many different pictures of the Ideal World graph. In fact, because of the inherent non-repeatability of the Real World, I can't draw the same picture twice. Requiring uniqueness of the Real World graph would make the concept vacuous. Observable variations in my Real World graphs of the same function usually come from using different scales on the axes and different degrees of care in how I follow the dots.

Regardless of the great number of Real World pictures that I can draw of the Ideal World graph of a function, I will call any picture I draw 'the' graph of the function. I do this because everybody does. The fact that everybody does something is not generally a good reason for doing it, but it is not a good reason for not doing it either. Presumably everybody knows that the picture on the paper is not really 'the' graph of the function, so there is no harm in calling it 'the' graph of the function.

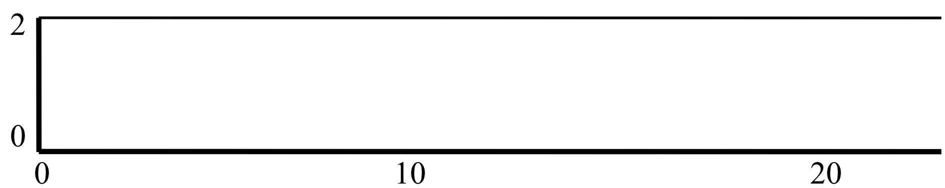
Personally, when I graph a function, I shun the use of dots as much as I can and just draw some curve that is 'kind of' like the graph. I do not even try to make the graph of a function the same each time I draw it, I just try to get the general shape right. Then I say, "This is <u>the</u> graph of the function, f. ".

I might add that I always try to draw a picture that looks good. A friend gave me a poster of one penguin talking to two other penguins and the caption read, "I cried because I had no shoes until I met a man who had no class." Functions deserve pictures with class, they deserve respect.

There is another function associated with uniform motion and that is the function *v* that gives the speed at every time. If the speed is 2 ft./sec. then *v*(t) = 2 for every value of t in the domain.

This is the graph of the function *v* that models the motion of the object.

So is



LECTURE 5-7

My search for the fireball led me into mysticism and astrology. Was this vision a sign that I was the next avatar? It seemed unlikely...

If I say that I can graph a function, I mean that I can draw a Real World picture that I think gives me a good idea of the general character and behavior of the Ideal World graph. If I say that I am going to graph a function, I mean that I am going to exercise the ability I spoke of in the previous sentence.

I have graphed the function that relates distance traveled to time for an object in uniform motion and I look around for another function to graph. Well, I have another function that I can graph, namely the function that models the distance a rock falls as a function of time. I'm going to call this function f and its rule is $f(t) = 16t^2$. This particular rock falls for two seconds so [0,2] is the domain of f.

The rock falls 64 feet in those 2 seconds and that seems like a lot to me; it's more than six stories. The range of this function is [0,64] and the domain is [0,2]. It does not seem possible to use the same scale on both axes because if I can see the [0,2] on the horizontal t-axis, I won't be able to fit the [0,64] vertically on the page. This means that my picture will be somewhat distorted and instead of drawing the picture I would like, I will have to settle for drawing the best picture I can. When I look at the picture, I want to get the feeling that distance increases a lot faster than time, even if the scales aren't the same.

The data looks like this:

time	distance
0.00	0.00
0.25	1.00
0.50	4.00
0.75	9.00
1.00	16.00
1.50	36.00
2.00	64.00

and the plotted points look like this

```
   |
 64|                  *
   |
   |
   |
   |
 48|
   |
   |
   |             *
   |
 32|
   |
   |
   |
   |
 16|        *
   |
   |       *
   |
   |    *
   |__*_______________________________________
   0        1         2
```

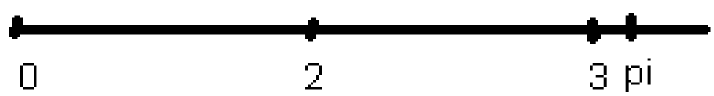
I have no curved ruler for these dots to fall on and I have to plot enough points so that I can fill in the picture of the graph freehand. I think that if I drew the curve I would get a feeling that distance is increasing faster than time. The fact that the curve ‘bows up’ seems to give the feeling that there is more ‘up’ happening than ‘over’.

I have considered three functions in particular but my remarks are valid for any function whose rule uses algebraic and trigonometric expressions. I can make some general comments about drawing the graphs of these functions.

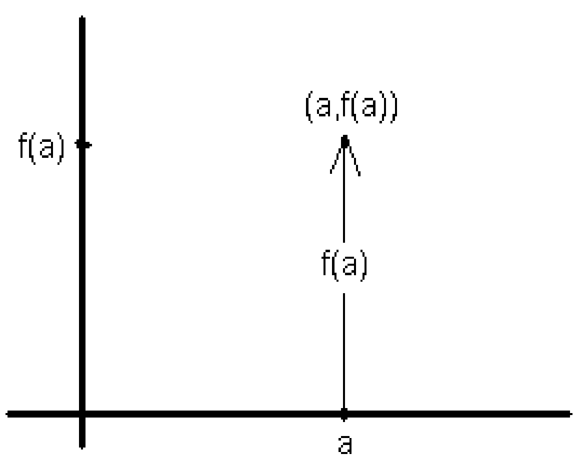
If I have such a function, *f*, in my pocket and I want to draw its graph, I start by drawing a horizontal and a vertical line on my paper. Usually I do it in that order. The position of the vertical axis depends on how many negative numbers are in the domain. I haven't yet had a function with negative numbers in the domain so my vertical axes have all been at the far left of the page.

I have no model in mind for this function, so I will call the independent variable *x*, the dependent variable *y*, and express the rule as y = *f*(x).

I use Real World scaled distances on my paper to represent the Ideal World numbers. If an axis was marked off in hundredths one could use a scaled length of 3.14 to represent π, but I don't recall ever using such an axis myself. What I actually do is make a mark a little bit bigger than 3 on the axis and call that mark π. I'm the artist. I say the length out to that mark is a picture of π, so, by God, it's a picture of π.



The signed distance of the point (a,*f*(a)) from the point 'a' on the horizontal axis is *f*(a) and I plot the points (a,*f*(a)) for as many values, a, in the domain as I can stomach. I measure the signed distance from the horizontal axis using the vertical axis as a ruler. Finally, I draw a smooth 'follow the dots' curve through the points and I have the Real World graph of *f*.



The signed distance of (a,f(a)) from 'a' on the horizontal axis is my visual experience of *f*(a). This is what the Real World graph gives me.

Suppose I know that the point (2,3) is on the Ideal World graph of *f*. What does that tell me? It tells me that *f*(2) = 3. If (a,b) is a point on the Ideal World graph of *f* , then 'a' is in the domain of *f* and b = *f*(a).

What does it mean for the point (2,3) to be on the Real World graph of *f*? If the graph is carefully drawn, it means that *f*(2) is pretty close to 3. If (a,b) is a point on a carefully drawn Real World graph of *f*, then 'b' should be pretty close to *f*(a).

I think about the picture as if it actually is the Ideal World graph, no matter how crudely I have drawn it. I use the expression 'graph of a function' for both the Real World and the Ideal World version and it is my opinion that I do this because I consider the picture of the Ideal World graph to actually be the Ideal World graph. I know it isn't true but I think it anyway.

When I look at a graph, the primary thing that catches my eye is the signed distance from 'a' to (a,*f*(a)), the signed distance between the graph and the horizontal axis. This is the picture of *f*(a), this is the picture of the output of the machine, the picture of the output of the function. I will describe the graph and the function by how this distance changes.

Usually there is some part of the domain where the function is particularly interesting to me and I want to get that in my picture. Domains of functions that model the Real World are inherently finite but a function might not be interesting on all of it. If I have an electric motor that runs at 1000 revolutions per minute and takes 2 seconds to get up to speed, then my domain of interest would be the first three or four seconds after I throw the switch. I want to see how it gets up to speed, not the hours it runs at 1000 rpm.

The second law of thermodynamics says that in the Real World, everything eventually becomes uninteresting. The speed of every motor eventually becomes zero and stays that way.

Since I can start my clock when I want to, I put my 'zero time' at an interesting time and I can put my 'zero spatial coordinate' at an interesting spot as well. As a matter of fact, 0 is usually an interesting point in the domain of any function, whether it models or not. This means that the origin of the coordinate system usually appears in my graph. I think that the only functions that are studied have graphs where the origin of the coordinate system is on the paper. 'Only' may be a little strong but the exceptions are few and far between. that are drawn on

Generally, I sketch the graph of a function as opposed to a careful, point by point, drawing that uses lots of points. I want the picture to give me intuition, not numbers. If I want a graph that I can use to evaluate the rule of a function, I will use a computer to draw it.

The sketch I draw will give me an understanding, say, of the behavior of some process as a function of time or of how the behavior of an abstract function depends on the independent variable.

I have been using the word ‘behavior’ a lot and I think I am using it correctly. After all, a function does ‘behave’. As the independent variable changes, the function changes in the sense that the values returned by the rule change. So the function is doing something as the independent variable changes; the function ‘behaves’. The word is not satisfying because I can go no further than saying, “The function behaves”. I have no language to describe that behavior so I can’t talk about it. The word ‘behavior’ just sits there unfulfilled. Clearly, I will have to develop such a language.

LECTURE 5-8

Success eluded me and I decided that my approach was too abstract. I needed the raw experiences of life; I had to learn to sing the blues. I returned to my job at Tom McAn's...

Generally, I think that the most convenient way to describe the behavior of a function is to sketch its graph. I define the properties of functions verbally and the graph allows me to see them. Every formal descriptive phrase about functions has a pictorial counterpart in the graph.

The graph, because it is a picture, gives me understanding that words cannot. Words come serially and the ideas transmitted must be held in buffers in my mind so that I can eventually consolidate them into the whole idea. My buffers are a little leaky and I often have to reread a page or have someone repeat what they have said. On the other hand, I 'see' the picture all at once. The whole idea is right in front of me to think about. When I understand something in its totality I say, "I see." There is a story I heard about a man watching a woman sculpt turtles. He asked her why she was doing this and she replied that if she could tell him why, she would write a book instead of sculpting. So it is with graphs. If I could say it, I wouldn't have to draw the picture.

I find the use of pictures in mathematics indispensable. There are people who can understand mathematics without pictures, but I am not one of them. There are mathematicians who do not have sight and I marvel at their intellectual power. There was a time when I tried to do mathematics without pictures, thinking that somehow there was a loss of honor their use. I remember working on a problem for three weeks using formal techniques and when I finally gave in and drew a picture, the solution was obvious. I became a believer in pictures on that day.

Of course I can't talk about the non-verbal information contained in the graph, but I can attach phrases to certain characteristics of the graph so that when I hear one of these phrases, the attached characteristic of the graph will come to mind as a picture. I can use these phases to talk about parts of the graph.

The 'characteristics' that I have in mind can all be described in terms of how f(x) changes with respect to how x changes.

The critical idea in the concept of the graph is that the signed distance of a point on the graph, (a,b), from the horizontal axis is equal to b = f(a). <u>The space between the dot on the paper at (a,b) and the horizontal axis is a picture of f(a)</u>. But more than the blunt fact of the size of f(a), there is the subtler property of how f(x) is changing as x moves through the number, a.

If I am driving my car at 10 mph and there is a brick wall in front of me, not only is the 10 mph of interest to me but also whether I am speeding up or slowing down. I am interested not only in the size of the speed but also in how it's changing.

The independent variable, x, changes causing f(x) to change, which causes the 'signed distance between the graph and the horizontal axis' to change. All the basic properties of functions are described in terms of how f(x) changes as the independent variable changes. I see each of these properties in how the 'signed distance from the graph to the horizontal axis' changes as the independent variable changes. When I talk about what the 'graph is doing' I mean what the 'signed distance between the graph and the horizontal axis' is doing.

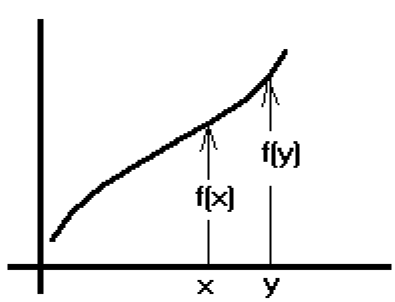
Since the properties that I want to consider have to do with what happens to the function when the independent variable changes, I'll look at the independent variable first. I have identified numbers with points on the horizontal axis, so I see the numbers in the domain of the function as points on the horizontal axis. I think of the independent variable 'moving' along the x-axis in the domain of the function.

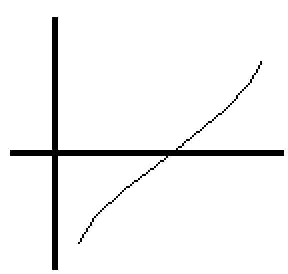
I am going to talk about x 'moving to the right' or 'moving to the left' . By 'moving to the right' I mean that x is taking successively larger numbers and not skipping any. If x moves to the right from 'a' to 'b', then x starts at 'a' and successively assumes all the values from 'a' to 'b'.

There are two such phrases that I look at now: "**a function is increasing**" and " **a function is decreasing**."

A function, f, is **increasing** on an interval [a,b] if whenever x and y are in [a,b] and x < y, then f(x) < f(y). This is the formal description. As x gets bigger, f(x) gets bigger, is a less formal description.

The visual, graphical description is





I describe the pictures verbally by saying that the graph is 'rising' or 'going up' as x moves to the right on the axis.

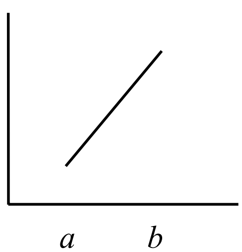
Since a graph that is rising as x moves to the right, is falling as x moves to the left, I must choose a direction for the movement of x in the definition of an increasing function. I have chosen to relate what f(x) does with movement to the right so that f(x) increases as x increases for an increasing function.

A **function is decreasing** if f(x) is decreasing as x increases or moves to the right.

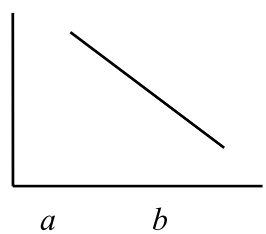
I can also let x move to the left and often will, but if I say that a function f is increasing or decreasing on an interval, I will always mean that f(x) is increasing or decreasing as x moves to the right.

If the function f is increasing or decreasing in an interval then so is its graph in the sense that the signed distance between the graph and the horizontal axis is increasing or decreasing. The adjective that describes what f(x) is doing on an interval also describes what the graph is doing over that interval and *vice versa*.

Of course the graph isn't really 'doing' anything, it is just sitting there. I am thinking that as x moves, the point on the graph, (x,f(x)), also moves. If the function is increasing, then the point (x,f(x)) rises as x moves to the right. If the function is decreasing, the point (x,f(x)) falls as x moves to the right. What the graph is 'doing' is what the point (x,f(x)) is doing as x moves to the right. If I want to talk about what the graph is doing as x moves to the left, I will have to explicitly say that x is moving to the left.

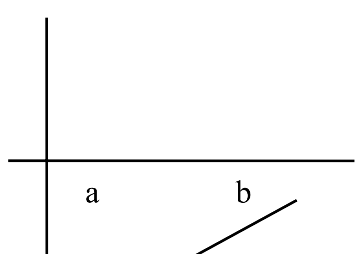


increasing

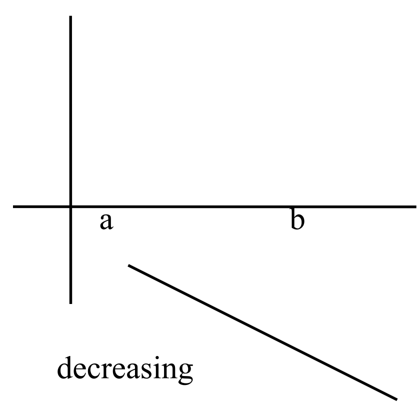


decreasing

Here are two more examples.



increasing



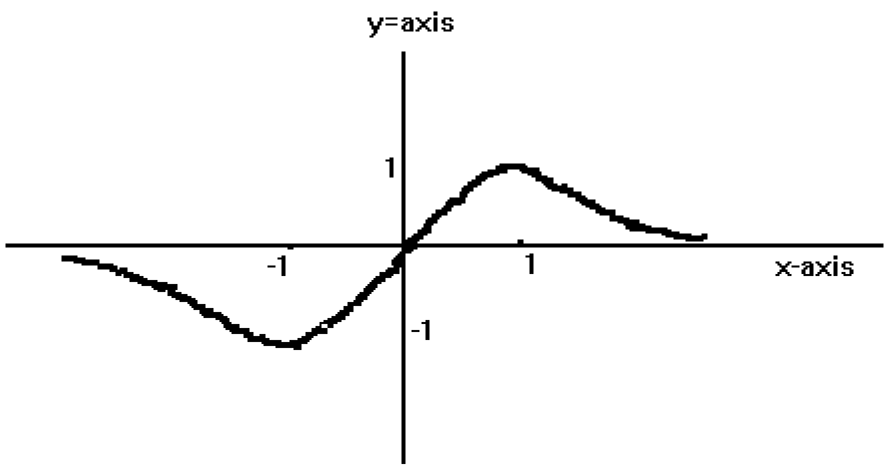
The graph can be increasing and getting closer to the horizontal axis, decreasing and getting further from the horizontal axis.

There aren't a lot of things that a function is able to do. It can increase, decrease, or be constant and that is about it. There must be more to the life of a function than this and there is. There is <u>how fast</u> the function is increasing or decreasing, but I will consider this later.

LECTURE 5-9

One day I sold some sandals to a man from a Buddhist Monastery. He too had met failure at achieving his life's ambition. On my way home from work I bought a racing form...

I am going to draw some graphs, that is, I'm going to draw some Real World pictures that are artistic renderings of the Ideal World graphs of functions. I am not concerned with what the rules of the functions actually are, I am only concerned with developing a language that I can use to describe the pictures.



The Real World graph I have drawn stops before it gets to the end of the positive x-axis, much less infinity and yet this picture implies that the Ideal World graph goes on to infinity in both directions. I stop drawing when the intention is clear; or when anything further I might draw would just be more of the same. The above picture might have meant that the domain of the function was the set of values of x that extended from one end of the Real World graph to the other, but in that case I probably would have put some mark on the axis showing the ends of the domain. Anyway, the domain is usually written down somewhere nearby and it is possible to check if there is doubt..

Interpreting the symbols of mathematics is something like reading Hebrew without the vowels; you have to know what it says before you can read it.

I think of the far left of the x-axis as going to $-\infty$ and the far right as going to ∞. I think that ∞ and $-\infty$ are in the Ideal World but they are not <u>on</u> the x-axis and can't be reached by going out on the x-axis. The point, x, can only get closer to these infinities as it moves out the x-axis to the right or left.

I see x coming in from $-\infty$, moving down the axis from the far left to the far right. As x comes in from $-\infty$, the graph is decreasing until x gets to -1, and then the graph increases until x = 1. From there on out toward ∞, the graph decreases.
Since I say that the graph is increasing or decreasing depending on whether 'the signed distance from the point x on the horizontal axis to the point (x,f(x))' is increasing or decreasing, and since f(x) is 'the signed distance from the point x on the horizontal axis to the point (x,f(x))', the previous description of the graph is exactly the same as saying that as x comes in from $-\infty$, f(x) decreases until x = -1, then increases until x = 1, and then decreases as x moves on out to ∞ .

Sometimes the graph increases, and at other times it decreases. Well, 'time' doesn't have anything to do with it. The expressions "sometimes" and "other times" are just figures of speech. What I mean is that for some values of x the graph is decreasing and that for other values of x, the graph is increasing. I suppose that when I think of x moving, I think of the movement happening through time and so I use expressions involving time.

At x = -1 and x = 1 the graph is neither increasing nor decreasing and the behavior at these points needs another name. I will say that the **graph is flat** at these points.

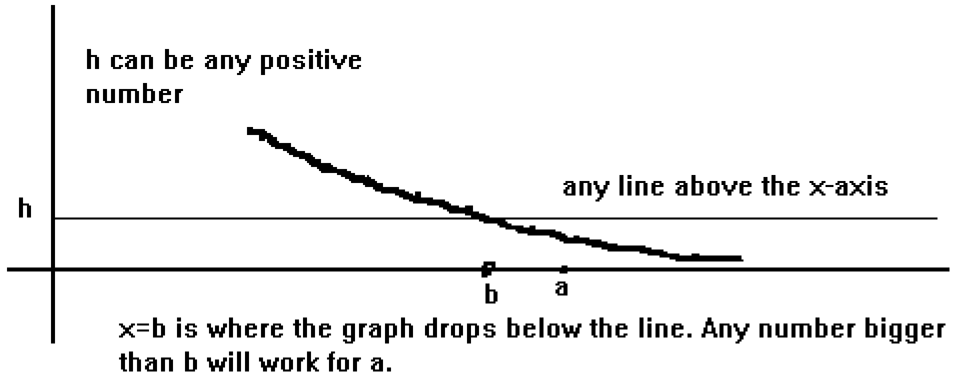
It isn't flat for very long, or perhaps I should say 'for very far', but a single point is long enough to deserve a name. I don't know that 'flat' is a generally accepted name for these points but in my opinion it is a good one. I will also say that the **function is flat** at these points and that **f(x) is flat** at these points.

The points x = -1 and x = 1 are special for another related reason. The number, f(1) = 1, is the largest that f(x) gets and, f(-1) = -1, is the smallest. f(1) is called the maximum of the function or sometimes the 'global' maximum. The word 'global' is used because it is the maximum of f(x) as x takes values from the entire domain. The number f(-1) is the global minimum of the function.

Not only is f(x) decreasing as x goes to infinity, f(x) is decreasing to 0. As x goes to infinity, but never getting there, f(x) gets close to 0, but never gets there. I will often use the expression **'f(x) gets arbitrarily close to 0'** to emphasize the fact that <u>f(x) doesn't stop short of 0</u> as x goes to infinity. If 'h' is any number bigger than zero, no matter how small, then as x gets large, f(x) eventually gets closer to zero than 'h'.

I can say all of this in terms of the graph. The signed distance from the horizontal axis to the graph goes to zero as x goes to infinity. Usually I leave out the 'signed distance' stuff and just say that the graph goes to 0 as x goes to infinity or the graph gets arbitrarily close to 0 as x goes to infinity. I use f(x) and 'the graph of f ' interchangeably.

The idea of the graph getting arbitrarily close to 0 as x goes to infinity can be expressed in a picture. If I draw <u>any</u> horizontal line <u>above</u> the x-axis, no matter how close the line is to the x-axis, there is some value of x, say, x = a, where the graph lies between the horizontal axis and the line for all values of x larger than 'a'. A lot of a's will work.

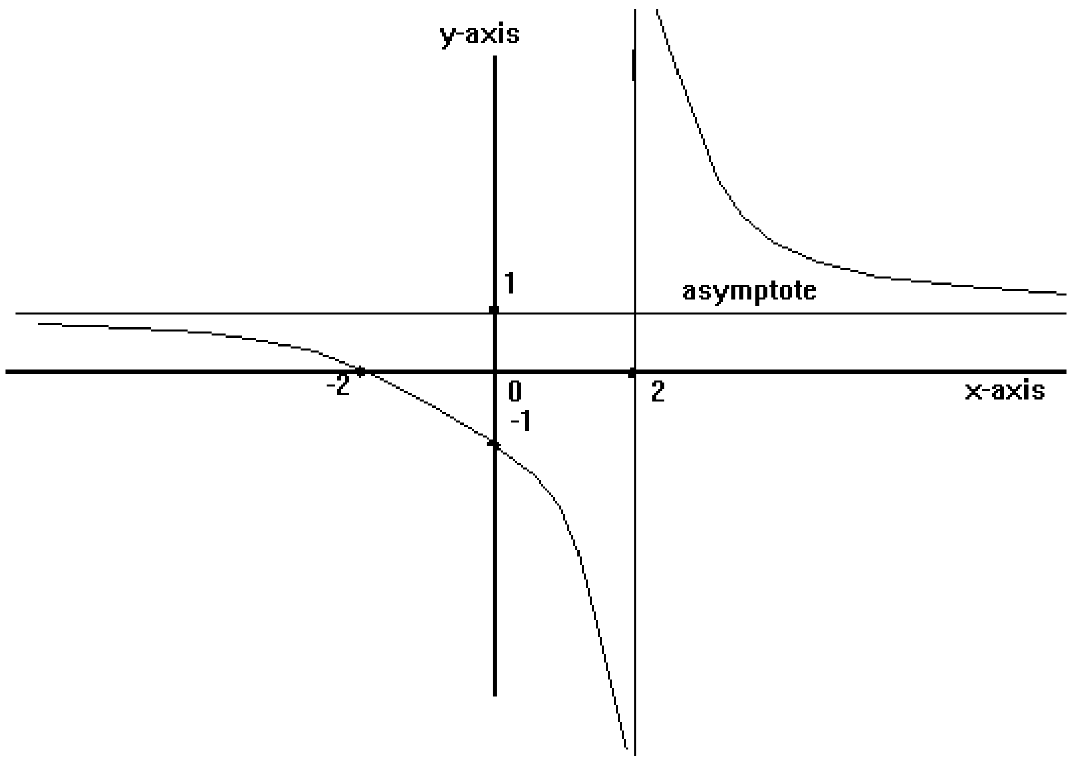


Another word used to described the behavior of the graph as x goes to infinity is '**asymptotic**'. The graph is asymptotic to the x-axis as x gets large. The x-axis is an **asymptote** of the graph.

When I say, "...as x goes to infinity...". I am thinking that in the Ideal World, x is taking every number in increasing order along the horizontal axis. The phrase implies that x gets arbitrarily large, meaning that x gets larger than any positive number. I can write the phrase symbolically, "...as $x \to \infty$...". I might 'make' or ' let ' x equal every number in increasing order along the horizontal axis if I felt that I had some role to play in x getting these values. I also say "...as x gets large..." to express this idea.

The expression "...as x goes to -infinity..." means that x takes successively smaller numbers all the way down the horizontal axis to the left. I write this symbolically , "...as $x \to -\infty$...". I can use "...as x gets small..." but I find this a little un-natural since it is natural for me to think of something small as being close to zero. If I think about it, though, -1 is less than 0 and so -1 is smaller than 0. Similarly, -2 is smaller than -1, so as x goes to
-infinity it is getting smaller and the phrase is correct. Be that as it may, I don't use "...as x gets small..." very much.

Here is another graph to talk about.

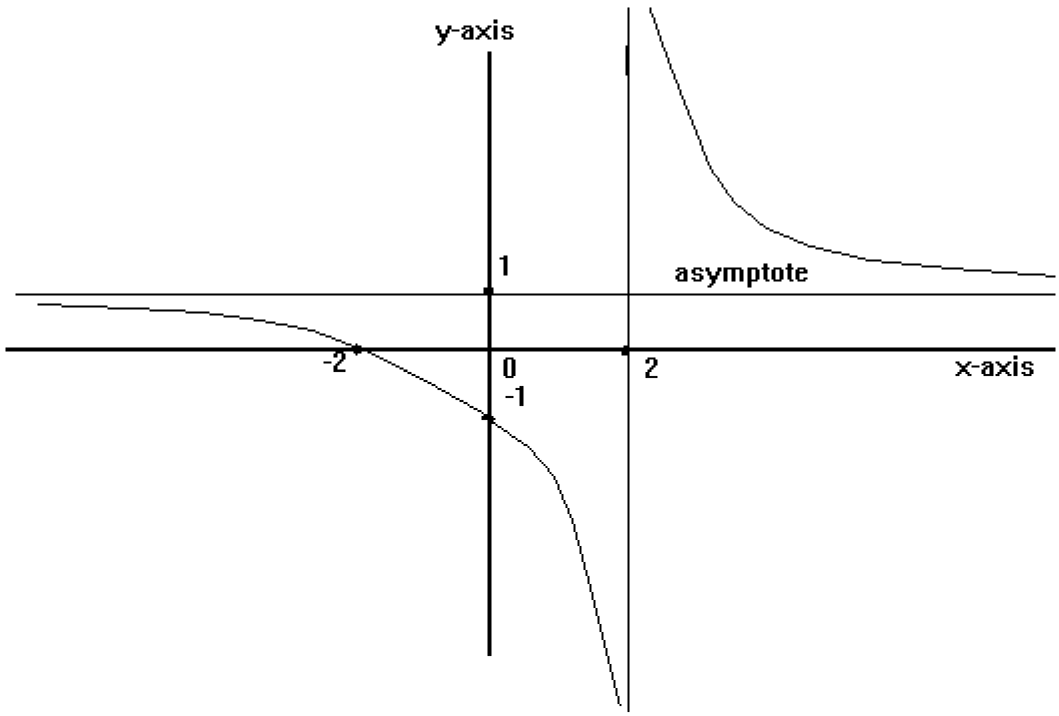


The horizontal line just above the x-axis is not part of the graph of the function, it is an auxiliary line that helps me talk about the graph. Since the y-coordinate of every point on the line is 1 and since every point whose y-coordinate is 1 is on the line, I will call this line ‘y = 1’.

Since every point on the x-axis has y-coordinate 0 and since every point with y-coordinate 0 is on the x-axis, I could call the x-axis, the line ‘y = 0’. I have to be a little careful with this language. If I just say ‘y = 0’, I am saying that the dependent variable takes the value 0. If I want the whole line, I have to say, “the line ‘y = 0’”.

As x gets large, the graph of the function approaches the line ‘y = 1’ from above; or in terms of f(x), as x gets large, f(x) goes to 1 through numbers larger than 1, or, f(x) gets close to 1 from above,. One-more-time. The graph of the function is asymptotic to the line y = 1 and the line y = 1 is a horizontal asymptote of the graph.

As x goes to $-\infty$ the graph of the function is asymptotic to the line y = 1 from below. I can just as well say that f(x) approaches 1 from below as x goes to $-\infty$.



The graph intercepts the x-axis at x = -2. This means that f(-2) = 0, which is usually expressed by saying that the function has a **zero** or **root** at x = -2.

The vertical line through the point x = 2 on the x-axis is not part of the graph either, but is another asymptote. The x-coordinate of every point on this line is 2 and every point whose
x-coordinate is 2 lies on this line, so, I call this line 'x = 2'. The line 'x = 2' is called a **vertical asymptote**. As x goes to 2 from either side, the graph gets arbitrarily close to the line x = 2, but in quite different ways. As x goes to 2 from the right, f(x) goes to infinity or, as the graph gets arbitrarily close to the asymptote from the right, it goes up to infinity. But when x goes to 2 from the left, f(x) goes to $-\infty$ or, as the graph approaches the asymptote from the left it, goes down to $-\infty$. The graph has a terrible discontinuity at x = 2. While
x = 2 is a critical point for the graph, <u>it is not in the domain of the function</u> and no point of the graph lies above or below the point x = 2. A graph cannot cross a <u>vertical</u> asymptote

It is certainly not rare for graphs to get arbitrarily close to a line as x goes somewhere. In such cases the graph is said to be **asymptotic** to the line and the line is called an **asymptote** of the graph. It is not, perhaps, the name I would have chosen if it were up to me, but it wasn't.

I say that the function ‘blows up’ to infinity as x goes to 2 from the right and ‘blows up’ to $-\infty$ as x goes to 2 from the left. This is not said by everybody but I think it is descriptive.

LECTURE 5-10

The racing form changed my life. I hit five exactas and seven daily doubles in a row. I invested my winnings in real estate and high yield junk bonds, all of which were successful...

I would like to step aside for a moment to say a few things about some of the expressions I am using, in particular, expressions like, “x gets close to 2” or “x goes to 2” .

First, I want to say what I mean by ‘close’. “I feel close to my wife.” and “We are close to an agreement.” are not the usages I have in mind. I am thinking more along the lines of physical separation. I might tell a friend, “If you are close to my house, stop in.” I mean, “If you are in my neighborhood, stop in.” How big is my neighborhood? If we live in the same town, my neighborhood is four or five blocks in any direction. If my friend lives in New York, my neighborhood is Tucson, which is where I live. If my friend lives in Berlin, my neighborhood is Arizona. ‘Close’ means that you are in my neighborhood.

A neighborhood of a point, a, on the x-axis is any interval, (c,d), that contains ‘a’, or what is the same thing, any interval, (c,d), where $c < a < d$.

When I say that **x gets close to ‘a’**, I mean that x is moving toward ‘a’, that x eventually gets inside and stays inside every neighborhood of ‘a’, no matter how small the neighborhood is, but that x NEVER equals ‘a’. Never. Never ever.

I will also say, **“...as x goes to ‘a’...”** and a lot of other things that have the same general meaning, instead of , “...as x gets close to ‘a’...” . I can write “x goes to ‘a’” symbolically by “ $x \to a$”.

I think of “...x gets large...” or “...x goes to infinity...” as being the same as “...x gets close to infinity ...”. There is a difference, however, between getting close to an infinity and getting close to a number like, say, 2. This is because ∞ and $-\infty$ have only one side and I can neither get close to ∞ from above nor get close to $-\infty$ from below. I can get close to 2 from both below and above, that is, from both the left and the right.

I use the expression, “...x goes to ‘a’ from the left...”, when x takes successively increasing values that are less than ‘a’ and getting arbitrarily close to ‘a’without equaling it. How close does x get to ‘a’? Well, it gets closer than any number less than ‘a’.

I say, “...as x goes to ‘a’ from the right...”, when x takes successively decreasing values that are greater than ‘a’ and getting closer to ‘a’.

The expression, “$f(x) \to 4$ as $x \to 2$”, means that f(x) gets close to 4 as x goes to 2 from the right, from the left, from any which way.

I am making a fuss about two crucial aspects of the statement, " x gets close to 'a'":

1. That x gets arbitrarily close to 'a'.
2. That x never equals 'a'.

The whole point of looking at how x moves is to be able to relate the way x changes to the way f(x) changes and it is just a fact of life that the values taken by f(x) when x is close to 'a' can be quite different from f(a). Since my main concern is how f(x) changes as x moves closer to 'a', I must separate 'close to 'a'' from 'at 'a''.

I think of a conical volcano 1000 feet high with a one inch diameter crater that goes all the way to the bottom. Since one inch is about the size of a Real World point with this scale, I am going to model the height of the volcano by an Ideal World volcano whose crater is one point wide. I let x represent the distance from the center of the volcano and let f(x) be the height of the volcano at the distance, x, from the center.

There are two happenings that I am concerned with: what happens to at 0 and what happens as x gets close to 0.

As x goes to 0, f(x) gets close to 1000, but f(0) = 0. What f(x) does at x = 0 is entirely different from what f(x) does as x gets close to 0.

If f(x) does the same thing as x gets close to 'a' from both the left and the right, then it is appropriate to use, "...as x goes to 'a'...", in describing what f(x) does. If f(x) does something different as x gets close to 'a' from the left from what it does when x goes to 'a' from the right, I have to distinguish the cases. And of course what happens at 'a' is something else again.

Now it is quite possible that what f(x) does at x = 0 is the same as what f(x) does as x goes to 0. It could happen that f(0) = 4 and that f(x) gets close to 4 as $x \to 0$. If the function models some physical process at x = 0, then the principle of continuity comes into play and what happens at 0 is the same as what happens as x approaches 0. A function that models a physical process can't jump. This last remark is not quite always true. It is really hard to say something that is always true. A physical process may happen so fast that it appears to jump even though in actuality it is continuous. In this case a person might want to model with a discontinuous function. It is a matter of time scale. The case where the behaviors are the same is special enough to deserve a name.

If what f(x) does at 0 is the same as what f(x) does as x goes to 0, I will say that the function is **continuous** at x = 0.

If f(a) = b, (this is what f does at x = a), and f(x) gets close to b as x gets close to a, (this is what f does as x goes to a), then the function **f is continuous at x = a**.

A **continuous function** is continuous at every point in its domain.

As x goes to some number, say 3, I evaluate the rule, f(x), at the numbers x takes as it goes to 3 and see what f(x) does. If the function models a physical process, then f(x) usually gets close to f(3) as x gets close to 3. In most of the exceptional cases f(x) gets close to one number as x goes to 3 from the right and f(x) goes to some other number as x goes to 3 from the left. This jump in f(x) is not possible for the actual physical quantities that f models because the actual physical quantities change continuously by the 'principle of continuity'. However, I may decide to model the physical process by a function whose rule does have such a jump.

Suppose f models the speed of a small electric motor that runs at 500 rpm. When I turn it on, the motor seems to instantaneously attain 500 rpm. Actually the speed continuously rises to 500 rpm but it happens so quickly that it appears instantaneous.

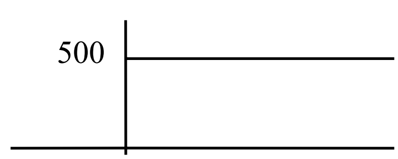
If I am interested in how the motor attains 500 rpm, which happens in about 1/100 second, I will scale the horizontal axis so that 0.01 takes up most of it. I will model the speed as a function of time by a continuous function f where f(t) increases smoothly from 0 to 500.

The speed of the motor can also be modeled by the function g whose domain is the interval [-1,3] and whose rule is given by

$$g(t) = 0 \qquad -1 \le t < 0,$$
$$g(t) = 500 \qquad 0 < t \le 3.$$
$$g(0) = 250.$$

I watched the motor for a second, turned it on and then watched for three more seconds. I'm not sure what to call g(0) since I don't really know how to think about the speed at t = 0. I thought about leaving g(0) undefined, but it is awkward to have a hole in the domain. I called it 250 for aesthetic reasons but I don't at the moment see why I couldn't have called it some other number. Since I don't know what g(0) is, leaving it undefined might have been the honest choice. Perhaps I'll do it that way next time.

As t goes to zero from the left, g(t) goes to 0. Actually g(t) = 0 for values of t < 0, so it certainly could be said that g(t) goes to 0. The value of g(t) is always 500 for values of t > 0, so g(t) goes to 500 as t goes to 0 from the right. The graph of g follows:



The function g is not continuous at t = 0. If a function is not continuous at a point, then it is **discontinuous** at the point.

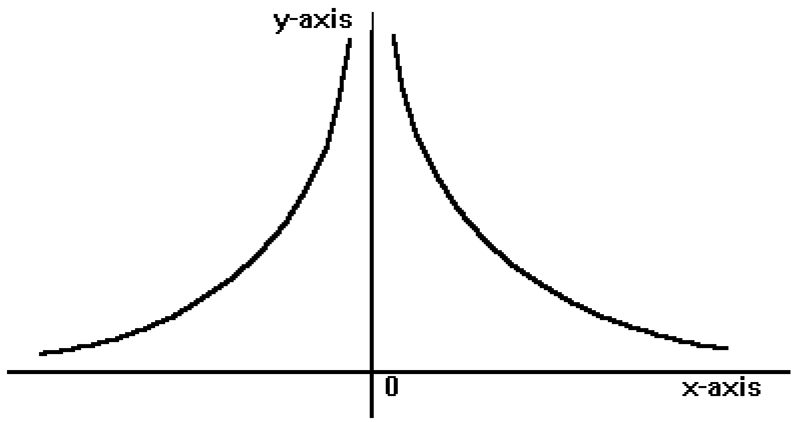
Even though the actual physical process is continuous at t = 0, the function that models it is not. The function I use depends on where my interest lies.

There are some other cases where f(x) gets large or small without bound as x goes to 3 and sometimes f(x) just wanders around aimlessly and doesn't get close to anything at all.

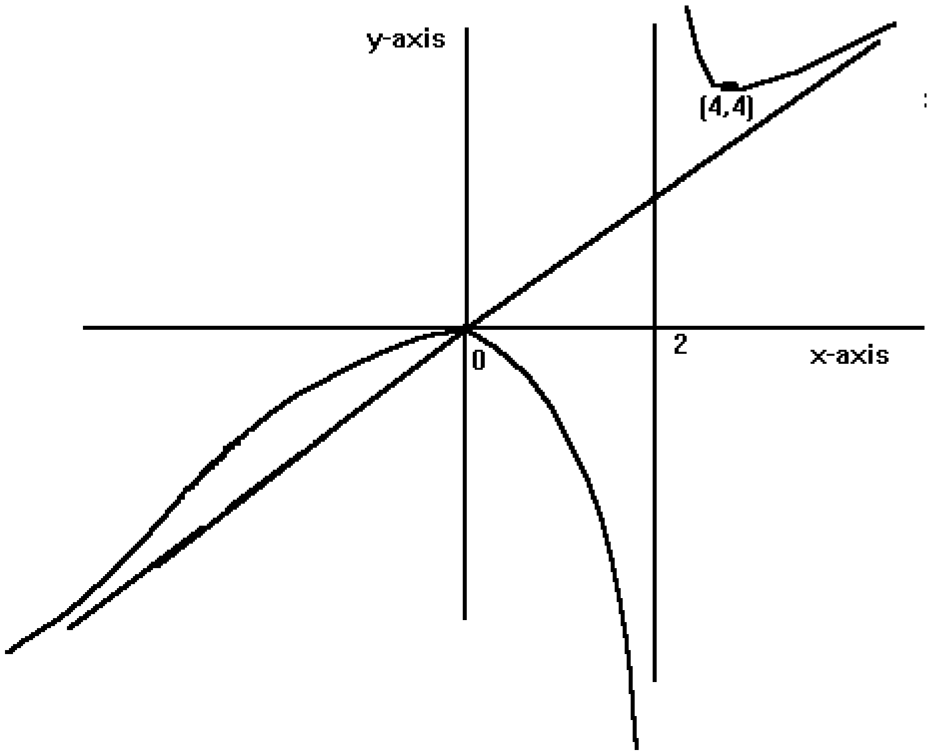
LECTURE 5-11

I now live in Laguna Beach in a house once owned by a Russian princess and I don't think about the fireball anymore...

The graph in this picture gets close to the y-axis as x gets close to 0 from either side but in this case it gets close to the positive part of the y-axis in both cases. I still call the y-axis or the line 'x = 0' a vertical asymptote. The point x = 0 is not in the domain of the function since the rule of the function is not defined at 0. There is no point on the graph over the origin.



A graph doesn't have to get close to both ends of a line in order for the line to be called an asymptote. The graph gets arbitrarily close to the x-axis from above as x goes to ∞ and, as x goes to $-\infty$ so the x-axis or the line 'y = 0' is a horizontal asymptote. The graph is increasing for negative values of x and is decreasing for positive values of x. The same can be said of f(x). As x goes to 0 from either side, f(x) gets arbitrarily large and in this case I might say that f(x) and the graph blow up to infinity as x goes to zero.



In this picture the graph also gets arbitrarily close to two lines and has two asymptotes. It has a vertical asymptote at x = 2. In this case f(x) goes to ∞ as x goes to 2 from the right, and goes to $-\infty$ as x goes to 2 from the left. The second auxiliary line is neither vertical nor horizontal. The graph gets close to this line from above as x goes to ∞ and as x goes to $-\infty$. I call this line an **oblique asymptote.**

The coordinates of every point on the oblique asymptote are equal and every point whose coordinates are equal lies on this line. It may not look like it, but I drew the line and I ought to know. I am exercising artistic license. If I say that the coordinates of the points on the line are same, then they are, regardless of what the picture looks like. Since the x and y coordinates of the line are equal, I call the line ‘x = y’.

There are two points on the graph that are of special interest, the points (0,0) and (3,4). The point (0,0) is interesting for two reasons. The first is that it is a zero of the function and the graph hits the x-axis there. The other is that the function is increasing just to the left of x = 0 and is decreasing just to the right of x = 0. The graph is flat at x = 0. Further more, the point (0,0) is the highest the graph gets and 0 is the largest that f(x) gets, for x in the interval (-2,2).
The graph is higher for values of x other than x = 0 and f(x) is bigger for other values of x, but the point (0,0) looks like some kind of a maximum. It looks like the top of a hill.

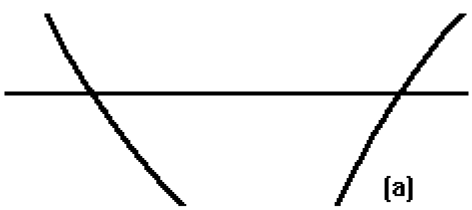
This point is called a **local maximum** of the function because it is a maximum for values of x in the locality of x = 0. Local maxima occur at the tops of hills on the graph. I can be a little more formal and say that f(x) $\leq$ f(0) for all x in (-2,2). This is the same local maximum that appeared in the function that modeled the position of the leaf that fell on that windy day last fall. Now I have a picture of it.

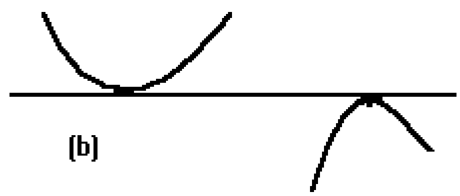
Similarly, the point (3,4) is a local minimum of the function. The value 4 is not the least value that f(x) takes but it is the least value when x is in the neighborhood or locality of x = 3. f(x) $\geq$ f(3) for all x in (2,4).

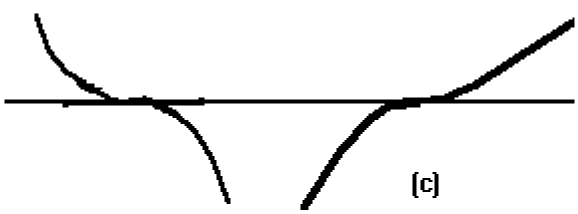
I can not help but notice that the graph is flat at both the local maximum and the local minimum. The graph looks like the top of a hill at a local maximum and the bottom of a valley at a local minimum and the graph is flat at both those points. The function has no global maximum or minimum in this example. There is no number in the domain where the function is greatest or smallest.

The points where f(x) = 0 are of special interest. If the function is modeling a physical process, f(x) = 0 means that some physical quantity is 0. If f(x) represents how far an object is from its starting point, f(x) = 0 means that the object has returned to the beginning. If f(x) represents the speed of an object, f(x) = 0 means that the object has stopped, even if not for very long.

There are only a few ways that the graph of a function can intercept the horizontal axis.

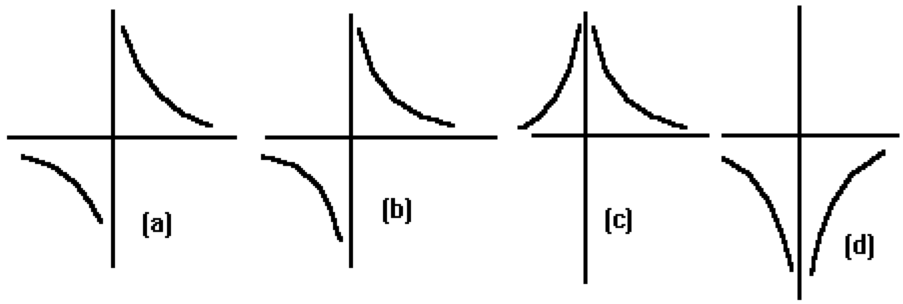






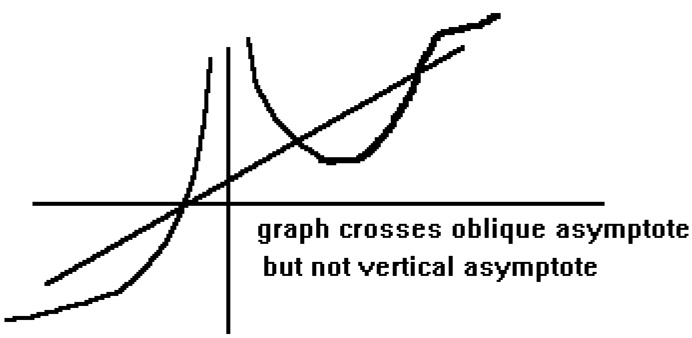
In (a) and (c) the graph 'passes through' the horizontal axis and in (b) the graph 'bounces back'. These six pictures show the only ways that a graph can intercept the horizontal axis. I think of the intercepts in (a) and (c) as 'pass through zeros' and the intercepts in (b) as'bounce back zeros'.

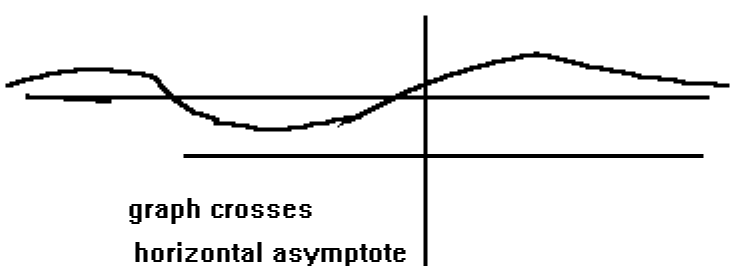
There are four ways that a graph can blow up.



In (a) and (b) the parts of the graph on opposite sides of the vertical asymptote are on opposite sides of the horizontal axis. As x moves through the critical point, the graph goes to infinity on one side of the horizontal axis and comes back from the other. In terms of the function, f(x) is one sign on the left side of the vertical asymptote and of the opposite sign on the right. In (c) and (d), f(x) has the same sign on both sides of the vertical asymptote and the parts of the graph on opposite sides of the vertical asymptote are on the same side of the horizontal axis. The graph comes back from infinity on the same side of the horizontal axis that it went to infinity.

If the line x = a is a vertical asymptote, then the point x = a is not in the domain of the function. The rule assigns no value to x = a. This means that the graph can't cross a vertical asymptote. On the other hand, a graph can cross a horizontal or oblique asymptote as many times as it wants.





These are perfectly good graphs, more or less.

LECTURE 5-12

I read the other day that a large fireball had struck a hilltop monastery. The monk who had bought the sandals did not live there, however, and I felt relieved that such an improbable coincidence had not occurred...

The examples illustrate the fact that the way graphs look is fairly restricted. Another interesting restriction is that any vertical line can meet the graph in at most one place.

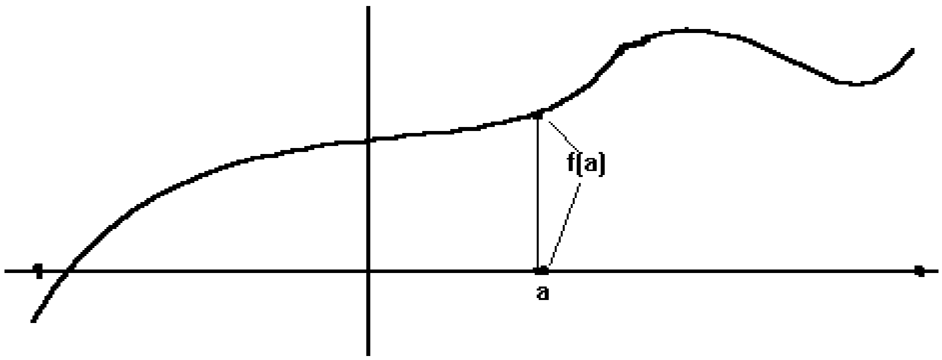
Every vertical line must hit the horizontal axis. The domain of the function is on the horizontal axis and there are there are just two possibilities: the vertical line hits the horizontal axis at a point of the domain or it does not. For the sake of argument I will suppose that the vertical line hits the horizontal axis at x = 6. I can refer to this line as the 'line x = 6' since a point lies on this line if and only if its x-coordinate is 6.

If 6 is not in the domain, then there is no value of f(6), so that there is no point (6,f(6)) on the graph and there is no point on the graph whose x-coordinate is 6.

If a point, (a,b), is on the graph of f, then a is in the domain of f and b = f(a). If the point, (a,b), is on the line 'x = 6', then a = 6. If two distinct points, (6,b) and (6,c), $b \neq c$, are on <u>both</u> the line <u>and</u> the graph, then f(6) = b and f(6) = c and there are two distinct values for the rule of f evaluated at x = 6. This is absurd because f is singled valued and there is only one number that f associates with 6.

A vertical line can meet the graph of a function at most once. This is commonly called the **vertical line test.** This means that any curve that does not pass the vertical line test can't be the graph of a function. A circle can't be the graph of a function, for example.

Countless times in my life I have drawn a coordinate system on a piece of paper, with a flourish drawn some curve on the coordinate system that passed the vertical line test, and said, “Consider the function, f.”



The curve goes up, it goes down, and since I draw it in one motion, it is continuous. If someone asks me to pick a number, I pick 6. If someone asks me to pick a function, I pick a function whose Real World graph looks like what I drew and think of it as representative of ordinary functions.

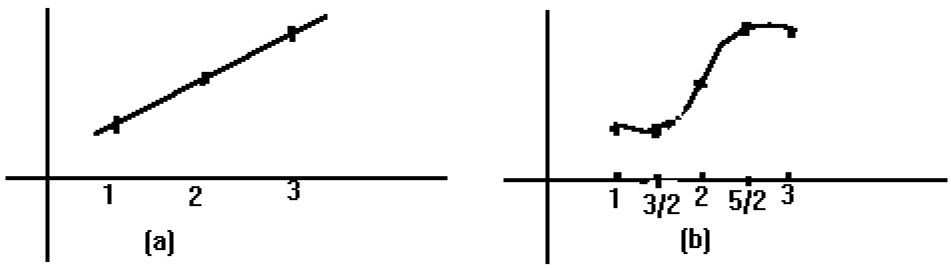
God only knows what the rule of this function is in the Ideal World; it is certainly beyond the reach of mortals. I do have, however, a way to evaluate the rule in the Real World. If the line ‘x = a’ hits the curve I have drawn, then ‘a’ lies in the domain of the function and the signed distance between the curve and point x = a is an approximation to f(a). It is an approximation because I have to measure the distance with a ruler. The function lies hidden behind the Real World graph.

This is quite different than having the rule explicitly, getting points by evaluating the rule at some values in its domain and sketching a graph. When I draw the picture of $g(x) = x^2$, I know what I’m drawing a picture of. This function is not hiding but is standing in front of me with its explicit rule in its hand for all the world to see.

Up through the 18th century, functions had explicitly given rules. In the 19th century the idea of function was broadened so that if a single valued rule existed implicitly, it was good enough. In the 20th century many functions that model physical processes do not have explicit rules. Data is obtained experimentally or mathematically and a finite number of points are plotted in a coordinate system. A curve is drawn following the dots and this is considered to be the graph of the function. The physical experiment allows me to evaluate the rule without knowing what it is and this allows me to draw the Real World graph of a function without knowing how the function generates it. It is a shadow in theReal World of an unknown Ideal World object.

The rule exists, it is just unknowable. It still assigns numbers in the domain to numbers in the range but in an ultimately obscure way; at least to humans. There is nothing illegal or immoral in this and the function may feel pleased that its rule is so deep.

When I obtain data and draw a graph by following the dots, I hope that graph is pretty close to what I would get if I knew the rule explicitly. This assumes that I have enough data points and this is sometimes not justified.



With just the three data points for x = 1, 2, and 3, I would be tempted to follow the dots and draw the line in (a). If I take data at the in-between points, I would have drawn the curve in (b). Now I have to wonder what kind of wiggles I might get if I take even more data points. Maybe none at all, maybe a lot. I am left in doubt. Paradise is lost.

If I have the experimental capability, I take as many data points as I need to convince myself that my graph is close to physical reality. Sometimes that capability is dear. I might want to find the relation between the diameter of a shaft in a gas turbine and the speed at which the turbine destroys itself. Each data point costs me a gas turbine. There could be a narrow range of diameters where the destruction speed is quite high and I would like to take the data points close enough together to find that range. The problem is that I don't know how narrow that range is or even if it exists. The applied mathematics problem has slipped over into economics.

Further Considerations of Chapter 5

Now that the graph of a function is defined, there is the problem of actually drawing it. Many functions are perceived through data taken from measuring instruments and in this case the data is plotted and a curve drawn through the resulting dots. This approach is also possible if the rule of the function is known. The rule is evaluated at a bunch of numbers, the points plotted and a curve drawn through them. It is reasonably effective if you have a lot of time or a computer but I have never found it much fun.

What I would like to consider is the problem of sketching the graph of a function if I know its rule and domain. I normally sketch the function over its natural domain, that is, the set of numbers where the rule is defined. This sketch gives the picture on any subset of the natural domain so I would have the graph of any function that uses this rule.

If I want to sketch the graph of a function, f, y = f(x), there are four questions that I consider. For a particular function, some of the questions may not be relevant, but I find that after I have answered all four, I can usually sketch the graph.

1. What does the function do as the independent variable gets large? That is, what does f(x) do as $x \to \infty$? What does the function do as the independent variable gets small? That is, what does f(x) do as $x \to -\infty$?

2. For which values of x does f(x) = 0?

3. For which values of x does f(x) “blow up”?

4. What is f(0)?

I will consider these questions one at a time.

1. What does the function do as the independent variable gets large? That is, what does f(x) do as $x \to \infty$? What does the function do as the independent variable gets small? That is, what does f(x) do as $x \to -\infty$?

This might seem like two questions to some people, but I think of them as one. The natural domain may not stretch to infinity in both directions, or in either direction for that matter, but even this gives information.

The domain of the function whose rule is $\sqrt{1-x^2}$ is the interval [-1,1], for example.

Be that as it may, there are plenty of functions whose domains do stretch to infinity.

There are not a lot of things that f(x) can do as x goes to infinity.

f(x) can approach some constant value. The speed of an electric motor starts at zero when it is turned on and approaches its nominal value as time goes to infinity.

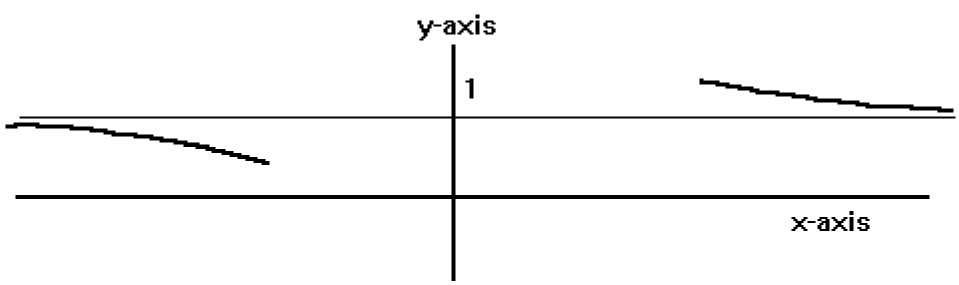
f(x) can endlessly repeat some pattern. A rock on the end of an ideal spring would bounce away endlessly, repeating the same motion over and over.

Finally, f(x) can become arbitrarily large. It, too, can go to infinity as x does. Money put in a bank that gives interest can grow to infinity if it is left there forever.

I'm going to look at a class of functions called **rational functions**. A rational function is the ratio of two polynomials and I am going to look at them because their graphs exemplify most of the kinds of things a graph can do.

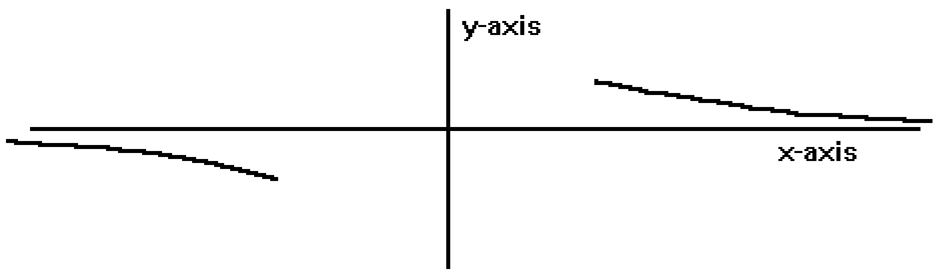
The natural domain of the function whose rule is f(x) = x /(x - 1) is the set of all numbers except 1 and the independent variable can go to infinity in either direction. As x becomes very large, the -1 in the denominator becomes insignificant and f(x) gets close to 1. Since the denominator is always less than the numerator, f(x) is always bigger than 1 and so approaches 1 from above. As x $\rightarrow -\infty$. the 1 still becomes insignificant, but now the denominator is always greater than the numerator and f(x) goes to 1 from below. I have now answered the first question for this function.

Here's how the graph looks for large values of x.



The natural domain of the function whose rule is $g(x) = x / x^2 + 1$ is the set of all numbers. As x gets large, the 1 in the denominator becomes insignificant and g(x) looks like 1/x which goes to zero. Since x is positive, g(x) goes to zero through positive values. As
$x \to -\infty$, g(x) goes to zero through negative values.

This function has the interesting property that g(-x) = -g(x). Such functions are called **odd**. The graph of an odd function for negative values of x is the same as the graph for positive values of x except that it is reflected about the horizontal axis. Here is how the graph of this function looks for large values of x.



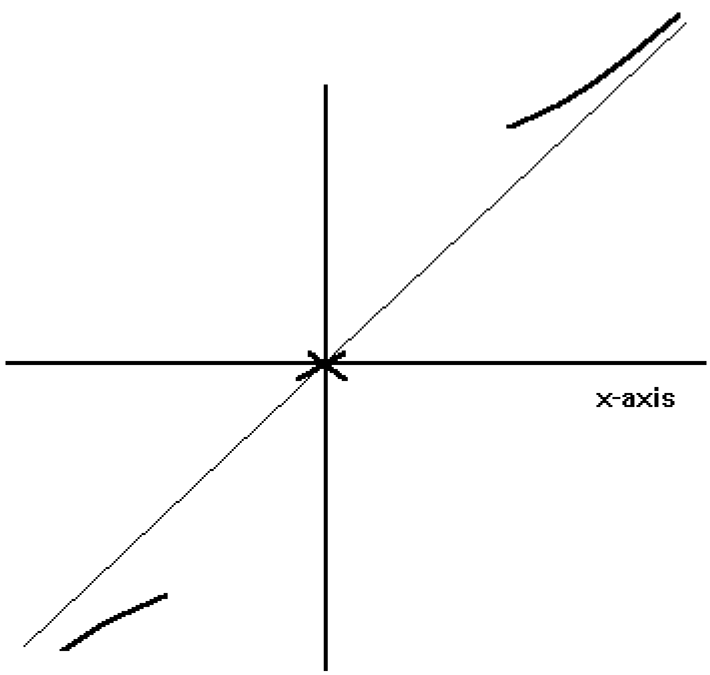
The natural domain of $h(x) = x^2 + 1 / x$ is the set of all non-zero numbers. As x gets large, the 1 in the numerator becomes insignificant and h(x) looks like x but is always a little bit bigger than x. This also true if x is negative and moving toward $-\infty$.

If the degree of the numerator of a rational function is larger than the degree of the denominator, it is a good idea to divide. In this case

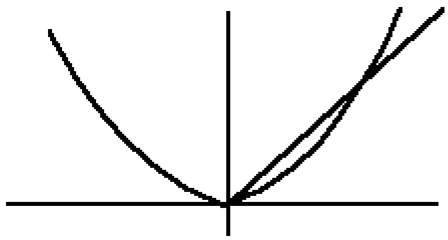
$$h(x) = x^2 + 1 / x = x + 1/x$$

and since 1/x goes to zero as $x \to \infty$ or as $x \to -\infty$, I again see that h(x) looks like x as x gets large in magnitude. The function h is also odd.

This is the graph of h(x) for large values of x.

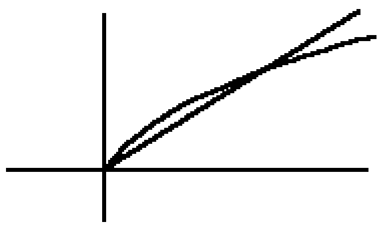


A function can grow faster than a line does as x gets large. The simplest example of this is the function whose rule is $f(x) = x^2$.



The graph of f will eventually rise above any line.

A function can grow slower than a line does as x gets large. An example of this is the function whose rule is $f(x) = x^{1/2}$. The domain of this function is the set of non-negative real numbers. It is not a rational function but it does grow slower than a line as x gets large.



Any line will eventually rise above the graph of f.

When x gets large some of the terms of an algebraicly expressed rule become insignificant and are dropped. When this is done the rule usually “looks like” some power of x. If the power is greater than 1, the function grows faster than a linedoes. If the power is less than 1, it grows slower than a linedoes. If the power equals 1, the function grows like a line.

2. For which values of x does $f(x) = 0$?

The rule of a function can return the value 0 for a value of the independent variable only if the numerator of the rule is zero for that value.

The function $f(x) = x /(x - 1)$ is zero only for $x = 0$, because that is the only place the numerator equals zero.

The function $g(x) = x / (x^2 + 1)$ is zero only for $x = 0$.

The function $h(x) = (x^2 + 1) / x$ is never zero because its numerator is never zero. Indeed, the numerator is always greater than or equal to 1. In this case, $f(x) = 0$ for no value of x.

The values of x where f(x) = 0 are the points where the graph of f meets the x-axis and I usually put a big dot on the x-axis at the zeros of f. The graph passes through these points.

3. For which values of x does f(x) "blow up"?

These are values of x that make the denominator zero. Since division by zero is not defined, a rule doesn't make sense at points where a denominator is zero and these values of x are <u>not</u> in the domain of the function. I put a big X on the x-axis at these points and draw a vertical line through the X. This is a vertical asymptote of the graph.

Why are the vertical lines through blow up points vertical asymptotes? I'll use f(x) as an example.

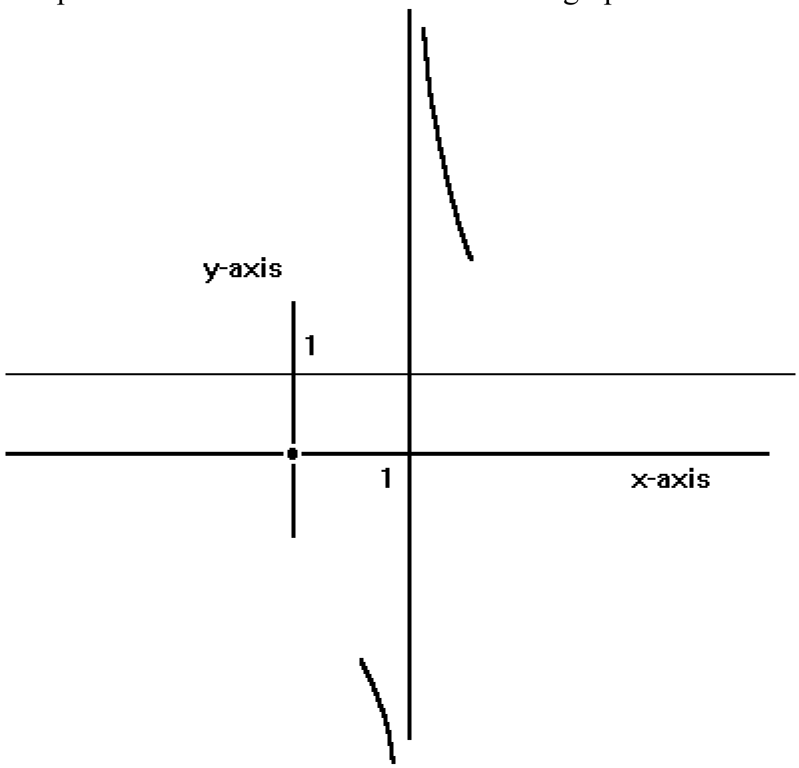
The denominator of f(x) = x / (x - 1) is zero only when x = 1. As x goes to 1, remembering that it never gets there, (x - 1) gets very close to zero. When I divide by something close to, but not equal to, zero, I get a very large number.

This is what happens generally, but the graph is very different on the opposite sides of 1. As x goes to zero from the right, the numerator is very close to 1, (x - 1) is a <u>positive</u> number very close to zero, and f(x) is a very large <u>positive</u> number, one might say very close to infinity. As x goes to 1 from the right, f(x) goes to infinity.

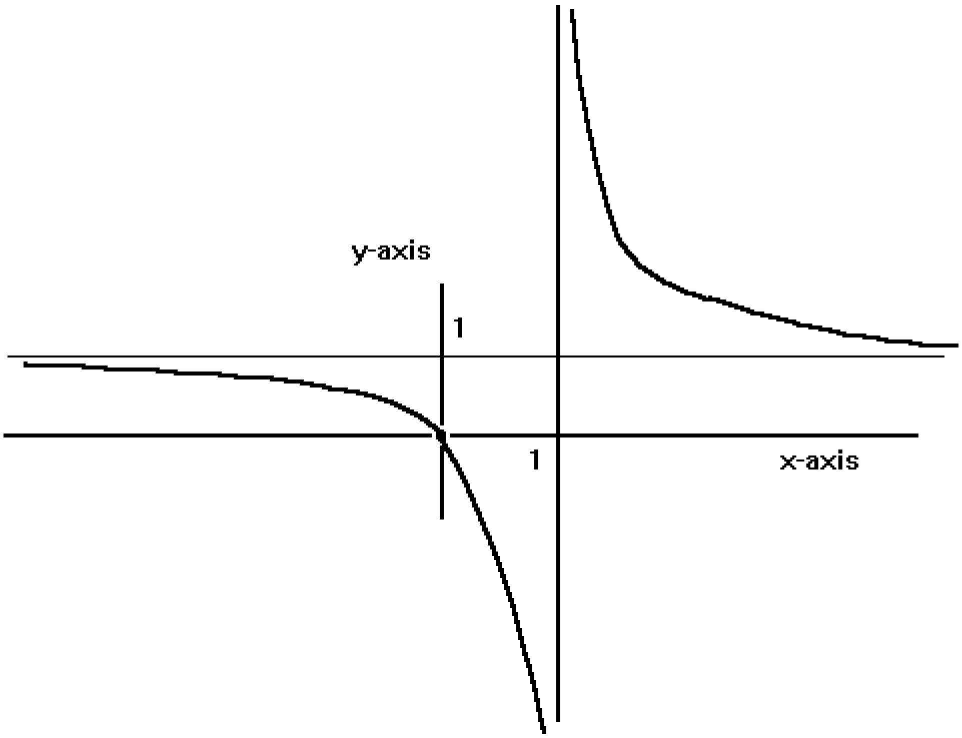
As x goes to zero from the left, the numerator is close to 1, (x - 1) is <u>negative</u> and close to zero, and f(x) goes to $-\infty$.

Since the numerator is close to 1, when I divide it by a small number, I get a big number. This is because I can subtract a small number a big number of times. By this 'small number' I mean a number close to zero. The only thing that can make a function blow up is a possible zero in the denominator of the rule.

The vertical line through the blow up point is a vertical asymptote of the graph of f. Here is a picture that shows the zero of f and the graph of f near x = 1.

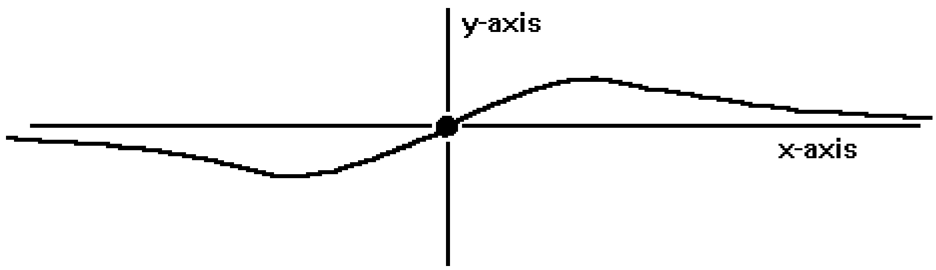


Finally, I put in both parts of the graph and connect them, making sure the graph goes through the big dot at x = 0 on the x-axis.

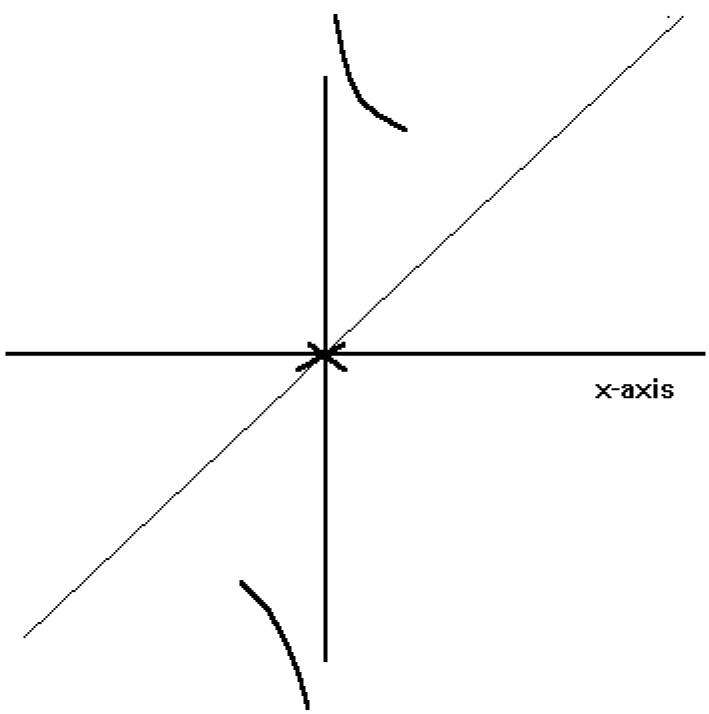

y-axis
1
1
x-axis

The denominator of $g(x) = x / (x^2 + 1)$ is never zero so g(x) never blows up. I need only connect the two parts making sure the graph passes through the origin since g(0) = 0.

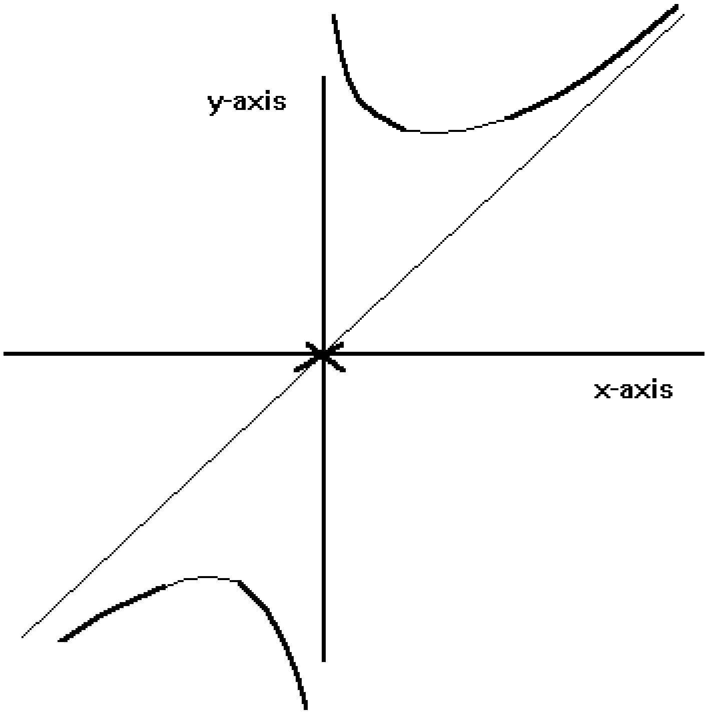
I connect the two parts of the graph I obtained looking at large values of x, making sure it passes through the origin.



The denominator of $h(x) = (x^2 + 1) / x$ is zero only when x = 0 so h(x) blows up only at x = 0 and the graph has a vertical asymptote at x = 0. As $x \to 0$ from the left, the numerator approaches 1 and $h(x) \to -\infty$. As $x \to 0$ from the right, the numerator approaches 1 and $h(x) \to \infty$. Here is the graph of h near the 'blow up' point, x = 0.



Here is the final sketch, combining the graph for large values of x with the graph for x near the blow up point, x = 0, and connecting the two parts.



4. What is f(0)? The answer to this question gives me where the graph meets the vertical axis. In the examples I'm looking at, I already know the answer and I have already sketched the graphs. I don't think of f(0) as a big deal but it is usually easy to calculate and it does tie down the graph at one point in a sketch where I am plotting pitifully few points. The only other points I plot are where the graph crosses the x-axis.

The functions, $f(x) = 2x / (x^2 + 1)$ and $f(x) = (x + 2) / (x - 2)$ have the graphs discussed in Lecture 9. The functions $f(x) = x^2 / 2(x - 2)$ and $f(x) = 1 / x^2$ have the graphs discussed in Lecture 11.

My general philosophy of graphing is to sketch the graph for x large ($x \to \infty$) and x small ($x \to -\infty$), sketch the graph near blow up points and zeros and then connect the parts.