CHAPTER 4

He was an average person in a uniform world.

LECTURE 4-1

As a child, mathematics was quite difficult for me. My classmates snickered when I gave my inevitably incorrect answer...

I think that if I drop a rock from anything that I can climb, I know how long it takes to fall a given distance and I know how far it falls in a given time. In the first case I model the position of the rock with a function, f, whose domain is the set of numbers from 0 to the number which represents the height from which I drop the rock, and whose rule is $t = f(s) = 1/4\sqrt{s}$. In the second case I model the position of the rock with a function h whose domain is the set of numbers from 0 to the number that represents the time it takes the rock to hit bottom, and whose rule is $s = h(t) = 16\,t^2$.

It is taking me too long to say these things and I need to streamline the language. As in the evolution of any language specific to a particular discipline, it tends to brevity. But brevity means that words are being left out, and a decision has to be made about what to keep and what to dump.

There are parts of any subject that everyone who is thinking about it or working on it knows. If I tell a mechanic to 'replace the lifters', that brief phrase encompasses all of the procedures necessary to fill several hours of labor. The phrase that I use to define or describe something may even be formally incorrect, but if everyone knows what I mean, that's OK. As time moves on, fads change and the language changes. New ideas come into fashion and old concepts are given new names in a new context. Anyway, I am going to try to make the language a little slicker.

There is one basic idea, function, which is a rule and a set. I want a reasonable way to talk about these things.

I'll see what I can do with the 'first case':

"In the first case I model the position of the rock with a function, f , whose domain is the set of numbers from 0 to the number which represents the height from which I drop the rock, and whose rule is $t = f(s) = 1/4\sqrt{s}$ ".

I think that anyone who is familiar with dropping a rock and the idea of function realizes that the domain must be 'the set of numbers from 0 to the number which represents the height from which I drop the rock'. I'm going to replace my 'first case' description with,

"In the first case I model the position of the rock by the function, f *,* given by $t = f(s) = 1/4\sqrt{s}$ ".

The truth is that when I think about function, I think about the rule first. This is because the domain is usually self evident and doesn't occupy much of my time. The rule, on the other hand, occupies a lot of my time. Even though I tend to neglect the domain, I do remember that the domain and the range are both necessary for the function to fulfill its appointed destiny. When I think about an automobile, I think about the electrical system last, but the car won't run without it. There may be something about the particular model I'm considering that causes the domain to be out of the ordinary, in which case I should specify it, but otherwise I will mention it only as the spirit moves me.

I have a friend who conjectured the following meta-theorem:

Given any mathematical idea, no matter how trivial, there is a problem about that idea that is arbitrarily hard. The idea of domain is no exception but 'strange' domains are rare when modeling the Real World.

A common way of expressing case one is:

"In the first case I model the position of the rock with the function $t = f(s) = 1/4\sqrt{s}$".

This statement is not correct. $f(s) = 1/4\sqrt{s}$ is an expression that represents a number, the number the rule gives when evaluated at s. The 't' is just the declaration of what the dependent variable is.

So $t = f(s) = 1/4\sqrt{s}$ is a symbol that represents a number, and certainly not a function, which is a rule and a set of numbers. One might argue that even if the symbols represent numbers, in their totality they represent the formula that gives the rule and since the rule of the function is what I think about most, it makes some sense to talk about the function, $t = f(s) = 1/4\sqrt{s}$. It is also short and sweet and everybody knows what it means. It is probably the most common way to express case one, although personally, I feel a little funny about it.

In fit of verbosity, I add a little to the last version.

"In the first case, I model the position of the rock with the function $t = f(s) = 1/4\sqrt{s}$, defined on the interval [0,16]".

In 1960 a function, like f , where <u>every</u> element in the range is the image of <u>exactly one</u> element in the domain, was called a **'one-to-one, onto function'.** In the 1990's it is called a **bijection**.

There is a synonym for function that I both like and find convenient and that is **mapping** or **map**. It is used primarily in the Ideal World and it comes up when I am thinking about the as function 'mapping' the domain onto the range. When I make a map in geography, I essentially associate every point in the geographical region with a unique point on my paper. I can say that I "map the geographical region" onto my paper. The rule maps the domain onto the range.

I could say that the function f given by $t = f(s) = 1/4\sqrt{s}$ maps [0,16] onto [0,1]. I could talk about the mapping of [0,16] onto [0,1] given by $t = f(s) = 1/4\sqrt{s}$.
Some years ago I was at a Blue Grass festival listening to some parking lot pickers with a friend, and a lick from the banjo player caught my friend's ear. The tune they were

playing was in the key of C and when it was over my friend asked the banjo player if the lick could be played in G. He tried it few times unsuccessfully and finally said, "I don't know. I'd have to see how it comes up in a song.” Context is everything. Whether a particular phrase will work depends on the context and functions come up in a lot of contexts.

If someone asks me if a particular phrase concerning functions will work, I tell them that I have to see how it comes up in a song, which, of course, they don't understand.

LECTURE 4-2

As I grew older I realized that financial and social success would be denied me if I did not remedy this flaw...

I now know the where and when of the rock and I want to look at the actual motion itself. By 'motion' I mean the property of the object that puts it at different places at different times. In the experiment where the rock fell for one second, my data shows that in each succeeding quarter second the rock falls further. If an object moves further in succeeding equal intervals of time, it is commonly said that the object is moving **faster**. In this case, I think of the motion of the object as always changing and if the motion of an object is always changing, I'm going to say that the object has **acceleration**.

Newton says that if an object has acceleration then there must be a force applied to it. The rock is moving faster so it has acceleration and must be experiencing a force. This force is supplied by the gravity of the earth.

Since the only way to change the motion of an object is to apply a force, my body can physically experience acceleration, or at least the force necessary to bring it about. When I ride in a car, the motion of my body is changing along with that of the car. As I accelerate away from a stop light, the back of the seat applies a force to my back and makes me go faster. When I come to a sudden stop, I feel the force of the dashboard on the hand I put up to keep me from hitting the windshield. When a person hits the street after falling from a twenty story building, their motion changes very quickly from very fast to zero. This is a very big change in motion in a short amount of time and requires a very large force, which the sidewalk supplies. It is the large force needed to decelerate this unfortunate person that breaks every bone in their body.

If there is no acceleration the motion is not changing and there is no force applied to the object. If I set the cruise control on a bus at 60 mph. the passengers can walk in the aisle with no problem since the motion is not changing and there is no force on them. If I put on the brakes while they are stretching their legs, their motion changes and they experience a force on their bodies.

Motion where **equal distances are traveled in equal times** is called **uniform motion**. This is a concept in both the Ideal World and the Real World. As opposed to the motion of the falling rock whose motion changes every instant, uniform motion never changes.

It is my approach to problems to look at the simplest cases first to get the lay of the land. The way the simple cases work out often leads me to how the general case works and this is going to be my approach here. I'm going to look at uniform motion first.

It is also true that I don't at the moment see how to proceed for the rock whose motion is always different, so I'll try something where I do see how to proceed.

I have decided to leave a general situation and go to a special case and from this special case I hope to see how the general situation works. There is something going on here. The big idea of mathematics and science is to find some general principle that is applied to special cases. The general principle is proved using contemporary methods and standards of proof and the validity of the general implies the validity of the particular. Supposedly, I should believe the special cases and examples because I have proved the general case. But it was the special cases and examples that led me to the general principle and belief in it in the first place.

Do I believe the special cases because I have proved the general case, or do I believe in the general case because of all the special cases that work? I end up with two kinds of belief. One, the intellectual belief that comes from seeing the proof of the general and the relation between the particular and the general. The other, the belief that comes from seeing examples and how these examples exemplify the truth of the general.

There are both Real World and Ideal World concepts that I believe in because of formal proof. A Real World example is the shape of a vibrating drum head. An Ideal World example is the irrationality of the square root of two. There are other concepts that I believe because I believe the special cases and examples. Even though I have seen a proof, it is the examples that make me <u>really</u> believe.

I suppose that the basic question here is, "Why do I believe that some things are true and not others." And what is truth? The formalist answer to this question is that there are axioms and rules of logic and whatever follows from the axioms using the rules of logic, is true. This is Ideal World truth.

In the Real World, truth is a little tougher. If I am going to use Ideal World truth in the Real World, then the axioms and rules of logic of the Ideal World must be true in the Real World. I can only determine if the axioms hold in the Real World by measuring and truth is only as true as my measuring instruments are accurate. Since they cannot measure exactly, truth remains behind the wall.

This would seem to leave the person working on Real World problems in some sort of philosophical fog, never really knowing what is true and what isn't. Since truth seems to change depending when and where you are in the universe, it seems questionable if 'Real World eternal truth' is even a meaningful concept. In the Real World, truth about the physical universe appears to be local in time and space.

The Real World cuts this Gordian knot by saying that truth is what works. There are not many successful contractors using non-Euclidean geometry. If it works better, it is more true. The Egyptians of the pharos apparently did not know why their formulas worked, only that they did. The lack of formal proof did not seem to adversely affect the quality of their construction. When I recall my decision to do the easy cases first, I also recall that the biggest pyramids are the oldest. It should also be noted that we don't build pyramids any more, we build space shuttles.

I really like this aspect of the Real World; solve the problem and get on with life. There is a story about a young mathematician that went to work for an aircraft company. It seems that when some airplane dropped a bomb there was a piece of chain, about 4 feet long, left dangling out the bomb bay doors and it beat up the side of the airplane. The young man was given the problem of weighting the chain so it wouldn't. After a couple of weeks of struggle trying to model the chain as a partial differential equation and with no success, one of the older hands came over to see what he was doing. Upon being told, his mentor suggested dropping the chain with the bomb. That is applied mathematics at work.

Knowing a formula and how to substitute numbers into it, is not enough for the person who applies mathematics. It has been my experience that the problem I have is not quite like the problem for which the formula was developed and this means that I have to alter it in some way so that it does fit. If I am going to alter it, I must know what each term in the formula means; mathematically so that I can predict how my changes will affect the outcome numerically, and physically because I must know which terms refer to which part of the physical system.

I once worked around some gas turbine people and there was little they did not know about gas turbines. If the turbine wasn't doing what they wanted, they knew what to change and by how much to make it right. They used rules of thumb or simple formulas which they had acquired over a long period of time by changing different parameters, observing what happened and remembering it. I could say that through extensive experimentation they had learned what each term in the formula meant. This is one path to understanding and in many situations it is still viable.

Another way to gain this understanding is by understanding the Ideal World origins of the formula. Most formulas come from the rules of functions that model physics. Most of these functions do not come out of the laboratory, as did the function that modeled the position of the rock, they are derived using Ideal World methods. The formula that is obtained in this way must be verified in the laboratory, but it came out of the Ideal World.

The pure mathematician can live in the Ideal World and never leave, but the person who is interested in applications not only has an office in both worlds but must establish a communication link between them. The position of the Ideal World mathematician is epitomized in Hardy's A Mathematician's Apology where he expresses the wish that his mathematics never be applied; not now, not ever. I find this point of view distasteful myself, but *degustibus non est disputandem.*

LECTURE 4-3

My attempts at romance were frustrated by my ineptitude at arithmetic and my childhood sweetheart eloped with a calculus instructor...

If I am going to use a model that has arisen through some Ideal World process, I need to have an *a priori* reason to think it's going to represent the Real World truly. While the model is going to be tested in the laboratory, it is probably rude to send the lab a stream of loony-toons models and I should at least believe my models are valid in the Ideal World.

I come to belief in the Real World through experiment and I might think some about belief in the Ideal World.

I believe that $\sqrt{2}$ is not a fraction in the Ideal World because I have seen the proof. Before the proof I had no intuition about the rationality or irrationality of the $\sqrt{2}$. There are numbers, like e^{π} , that have been known for hundreds of years and it is not known if they are fractions or not.

The Jordan Curve Theorem says that a simple closed curve has an inside and an outside. An example of a simple closed curve is a circle and the rest are just what I get from a circle if I stretch it and bend it, so long as it doesn't cross itself or break. I have never seen the proof of the theorem, but I use it without a second thought because it 'looks' so true.

The other day I was thinking about piles of bricks along the road to Santa Fe, although I can't quite remember what gave rise to the thought. I made a 'gedanken experiment'. On the left side of the road were 8735 piles and in each pile there was 5228 bricks. On the other side of the road there were 5228 piles and in each pile there was 8735 bricks. Which side has the most bricks? When I thought about the problem in terms of bricks, I wasn't sure. There were more piles on the left side, but there were more bricks in each pile on the right. There was nothing to do but count them. The number of bricks on the left side equals 5228 added up 8735 times. Fortunately, the arithmetic process of repeated addition has a name, multiplication, and a notation, 8735×5228. There are 8735×5228 bricks on the left. Then there should be 5228×8735 bricks on the right. My question is now answered by the Ideal World fact that $5228 \times 8735 = 8735 \times 5228$.

I believe there are the same number of bricks on each side of the road in the Real World because I believe that multiplication is commutative, $5228 \times 8735 = 8735 \times 5228$, in the Ideal World and that the Ideal World correctly models the Real World. I can't do this as a Real World experiment because I don't think that is it actually possible to make the piles, much less count the bricks.

Why do I believe that $5228 \times 8735 = 8735 \times 5228$? Because I believe the following:

$$1 \times 2 = 2 \times 1$$
$$2 \times 3 = 3 \times 2$$
$$2 \times 4 = 4 \times 2$$
$$3 \times 4 = 4 \times 3.$$

I can see why these are true. I can interpret them as areas and draw a picture that shows me why the order of multiplication doesn't matter. As soon as I see why the order of multiplication doesn't matter, I am willing to use it on all numbers; all integers anyway. What the hell, I'm easy. I think it is true for all numbers. I have gone from Real World examples to Ideal World truth and I assume this Ideal World truth reflects Real World truth.

I tend to think that if something is true for a lot of small positive integers, then it is true for all positive integers. Any number of people tell me that this is not a correct method of proof and they are absolutely right. But I am not talking about proof, I'm talking about convincing myself something is true. The reasoning that convinces me something is true may not be a proof and a proof may not really give me the feeling of belief.

LECTURE 4-4

And then one day, when things seemed darkest, I noticed how often the number 12 occurred. There were 12 inches in a foot, 12 months in a year, 12 eggs in a dozen...

I think that I am ready to model the motion of an object in uniform motion. If I were to jump right in and try to model the motion of the falling rock, I would have the function that models distance already at hand. I went to some trouble to get that function. Unfortunately, in the case of uniform motion, I don't have a function that models the distance moved by the object as a function of time, so I will do this first. Uniform motion is simple enough that I can get this function using reason and I don't have to go to the lab.

I'm going to suppose that the object moves 2 feet every second. In any second the object moves 2 feet and if the object moved 2 feet it took 1 second. Then in 0.5 seconds it moves 1 foot and in 0.25 seconds, it moves 0.5 feet. The following table shows the idea:

time in seconds	distance in feet
1.000	2.00
0.500	1.00
0.250	0.50
0.125	0.25
1.500	3.00
2.000	4.00

It appears that if I multiply the time by two, I get the distance the object travels. Since this is a ‘gedanken experiment’ in the Ideal World, I can skip the formula and go right to the function, which I am going to name S.

The domain of the function is the set of numbers that represents the interval of time when I watched the motion. It may have been moving before I looked at it and it may have continued moving after I left. I don't know. I wasn't there. I looked at this motion for 10 seconds and I started my clock at 0 when I began watching.

The domain of the function is [0,10]. The rule is to take a number from the domain and multiply it by 2. If I call the independent variable t and the dependent variable s, I can express the rule algebraically as

$$s = S(t) = 2t.$$

I have decided to use time as the independent variable and this is a bias on my part. I am thinking that I have a certain amount of time and I want to know how far can I go in that time. Drag racers look at it differently. They have a fixed distance and want to know how long it takes to travel that distance. They think of numbers like 5 seconds per quarter mile or time per distance. If physics had been developed by drag racers, my speedometer might read in minutes per mile.

If the object is in uniform motion and moves 0.5 feet in one second, then it travels two feet in 4 seconds and one foot in two seconds. A table of distances for a few values of time looks like this:

time in seconds	distance in feet
0.500	0.2500
0.125	0.0625
1.000	0.5000
1.500	0.7500
2.000	1.0000
4.000	2.0000

I wonder what this table looks like in fractions.

time in seconds	distance in feet
1/2	$1/4 = 1/2 \times 1/2$
1/8	$1/16 = 1/2 \times 1/8$
1	$1/2 = 1/2 \times 1$
3/2	$3/4 = 1/2 \times 3/2$
2	$1 = 1/2 \times 1$
4	$2 = 1/2 \times 4$

I can get the rule from either table but I find the table of fractions the easiest. The distance traveled is the time multiplied by 1/2. I should have said that the number representing distance equals the number representing time multiplied by 1/2, but that takes more time and I'm not getting any younger. I am going to call the number that represents time, time. The word ‘time’ now has two meaning. It is the physical concept and the number that represents it. I will use the same convention with other physical quantities. I will say ‘distance’ when I mean the number that represents distance as well as when I mean the physical concept.

I think that I will look at this second object for twenty seconds, during which time it moves 10 feet. I choose the name k for this function and its domain is [0,20]. The rule of this function is given algebraically by $s = k(t) = 1/2\ t = 0.50\ t$.

LECTURE 4-5

There were 12 signs of the zodiac and 12 legs on a spider. I had 12 fingers and 12 toes. There were 12 planets in the solar system. I couldn't remember how many siblings I had but I was sure that there must be 12 of us...

Now I will look for meaning in the individual terms of the algebraic expressions of the rules, s = k(t) = 1/2 t and s = S(t) = 2t . I know what the s and t mean, so I'll consider the third term.

The number '2' in the rule, s = 2t, is the same '2' in the expression, 'the object travels 2 feet in every second'. The '1/2' in the rule, k(t) = 1/2 t, is the same '1/2' as in the expression, 'the object travels 1/2 foot in every second'.

The 'distance the object travels in one second' times 'the number of seconds it travels' equals 'how far the object moved in that number of seconds'. The distance an object in uniform motion travels in one second is the critical number when it comes to describing the motion.

I am now ready to model the distance traveled by an object in uniform motion. I'm going to let *v* be the distance the object travels in one second and T the length of time I observed the motion. The watch is started when I first began my observation. If I let t be the time since the clock started, s be the distance traveled, and F be the name of the function that models, then the domain of F is [0,T], and the rule is

$$s = F(t) = v\, t.$$

One of the striking aspects of this statement of the model is how the language has changed. I say, "...let *v* be..." instead of, "...let *v* stand for the number that represents...".

The distance an object travels in a unit of time deserves a name and its name is **speed**. Speed only makes sense if the motion is uniform. If the motion is not uniform, speed is not defined and can't be used in the description of the motion. One of my major aims is to find a definition of speed if the motion is non-uniform.

I have been using feet and seconds as units of measurement. Of course the function doesn't know this. If I am being Ideal Worldly, I could say that the speed is 2. If I am in the Real World, I probably want to keep in mind what the units are so I would say that the speed is 2 feet in a second, or 2 feet per second, or 2 feet/second, or 2 ft./sec. depending on my mood. Other common units for speed are miles/hour, cm./sec. and meters/sec.

At every instant of time, the motion of an object in uniform motion is described by the speed, so I think speed must be at the heart of the function that models motion.

I now take a step forward and model the speed, *v*, of the object as a function of time. This would mean that *v* is the dependent variable and t remains the independent variable. The rule of this function associates time with the speed of the object at that time. Since speed captures the idea of my concept of what the word ‘motion’ means, I’m going to think of this function as the function that models motion.

The independent variable represents a readily measurable physical quantity that the function associates with the dependent variable. The dependent variable represents the physical quantity that has our interest, it is the ‘star of the show’. The dependent variable is the balloon that catches my eye, the function is the string that ties it to the ‘independent variable’ hook. If I know where the hook is I can find the balloon and the hook is easy to find.

I originally chose v as the symbol to represent the speed of an object and it seems reasonable to give that name to the dependent variable, which represents the speed as a function of time. But when I do that, I need another name for the actual speed of the object. v is the variable that assumes the values of the speed at different times. In the case of uniform motion, there is only one speed for v to assume and it needs a name. I choose σ. v is a variable that only assumes one value, which is different from σ, which is a constant.

I am also going to use v as the name of this function. Its domain is [0,T] and its rule is

$$v = v(t) = \sigma \text{ where } \sigma \text{ represents the particular speed of the object.}$$

There is only one value, σ, in the range of the function, so there is only one number , σ, that v can return. The rule returns the same monotonous σ for every number in the domain. I don't claim that v is the most exciting function I ever met, but when things get down and dirty and all the other functions have given up the fight, the constant functions and their relatives are there to fill the gap and push forward.

I have a self-inflicted ambiguity in using v for two very different things; the dependent variable, speed, and the name of the function. Context usually tells which one is meant and if there is doubt, I should say, “...the speed v...” or “...the function v...” . If it is not clear which interpretation is meant, it probably doesn't make any difference.

I use the same symbol for the dependent variable and the function because it helps me keep track of what the function is modeling. If I model position and motion with the functions f and g, $s = f(t)$ and $v = g(t)$, there is nothing to keep me from mixing up which function goes with which variable. If I call the functions s and v, I certainly won't forget which dependent variable they belong to. I might add that the double use of v is not as confusing as naming all your male children George.

I call distance s because that is the usual thing to do. I usually try to do the usual thing. The ideas of science and mathematics are hard enough to understand without adding the burden of translating a non-standard notation to something the reader is familiar with.

I chose v for speed because s was already used and velocity will someday replace speed as the quantity that characterizes motion.

I have a ‘thing’ about knowing what things are and distinguishing what they are from the symbols that represent them. I saw children's show on TV some years ago where a small boy, 7 plus or minus 2, was standing by a conveyer belt with various objects moving along it. The boy was looking for a dog. He picked up an object and the voice-over said, “Is this a dog ? No. This is a pineapple.” It wasn't a pineapple, it was a piece of rubber shaped and painted like a pineapple. Then there was a stuffed animal that was supposed to look like a cat and was called a cat, but it was not a cat. Finally he got the dog. “Is this a dog? Yes. This is a dog.” The child might have been confused, thinking that dogs were things that went bow-wow and slobbered on you, while the thing he had in his hand was something you kicked and threw at your sister.

Anyway, this program made me very sensitive to the use of ‘is’ and ‘are’. When I say that t is time, I hear the voice-over saying, “This is a dog.”, when it was a stuffed animal. But, I go ahead and say it anyway.

LECTURE 4-6

Every number must be 12. I was staggered by the simplicity and beauty of this idea. I was amazed that it had not been noticed before...

The idea of using speed to characterize uniform motion is just one instance of, in my opinion, a really big idea. The big idea is the idea of **density**.

If I am told that someone runs an hour before breakfast, I have no picture in my mind of their motion. If I want to picture the motion, I need distance as well as time. Covering one mile in that hour looks a lot different than covering ten miles.

If I am told that someone runs a mile before breakfast, I have no picture in my mind of their motion. I need time as well as distance to picture the motion. Running that mile in five minutes looks a lot different than running it in twenty minutes.

I need to know both the time and the distance an object moves to get a feeling for the way the uniform motion looks. Speed in an efficient way to give this information.

The number of automobile accidents in a city is not significant to me until I know the size of the city. Twenty five accidents a year would make New York City a safe place to drive but would make Pie Town, New Mexico a road safety disaster. Accident rates are given on the basis of accidents per thousand people, or accidents per thousand miles driven, or something like that.

If I want to describe how heavy a material is I say how much a unit volume weighs. If I want to describe how 'massive' a material is, I give the mass of a unit volume. I want to describe the properties of a material, not an object. The mass per unit volume of a material is commonly called the **density** of the material. Some densities of materials in kilograms per cubic meter are:

Interstellar space	10^{-20}
Laboratory vacuum	10^{-17}
Styrofoam	1.0×10^{2}
Water	1.0×10^{3}
Aluminum	2.7×10^{3}
Platinum	2.1×10^{4}
Central density of a white dwarf star	10^{12}
Uranium nucleus	10^{17}

These figures are not exact but give an idea of the magnitudes involved.

The mass in a unit volume is the same idea as the distance in a unit time. I can think of speed as the density of distance in time. If the speed is large, there is a lot of distance in a unit time. If a material is heavy, there is a lot of weight in a unit volume.

I am assuming that the material is homogeneous in the sense that every unit volume of the material has the same mass. This is the same as assuming that an object travels the same distance in any unit of time. The material being homogeneous corresponds to the motion being uniform.

If the material is some sort of mixture, then the 'massiveness' of the material is different at different places and this corresponds to non-uniform motion. In this case I have the same problem describing the material in terms of mass and volume that I have in describing non-uniform motion in terms of distance and time.

A lot of densities are called **rates,** which I presume comes from ratio. The super market sign, "Onions-3 lb. / $1.00" expresses the density of onions in a dollar. Postal rates give the density of mailing weight in money. Speed is a rate, as are most densities in time. Saying "rate of speed" is like saying "pizza pie" which means, "tomato pie pie", or so I have been told.

If I normalize a quantity by dividing by time, I think of the resulting density as a rate. If I have an object in uniform motion and I watch it for three seconds, there is only one way that three seconds can happen. But if I have a homogeneous material, there are a lot of ways to take 3 cubic inches of it. Time depends on one number, volume depends on three, length, height, and width. If the divisor depends on one number, I usually call the density a rate.

LECTURE 4-7

Based on my revolutionary idea, I opened a school for those people who, like myself, had found the mathematics taught in schools incomprehensible. In a few short weeks my students were ready to take their place in the modern world...

I am now ready to start chipping away at the description of non-uniform motion. I am going to think about the instant by instant description tomorrow and try to find a description of the motion of the trip as a whole.

It is exactly 60 miles from my Ideal World house to the Ideal World Santa Fe and I drive it in exactly 45 minutes. I can calculate my speed using the formula

speed = distance/time = 60 miles/ 3/4 hr.

Well, 60 miles in 3/4 hour means 20 miles in 1/4 hour, which means 80 miles in an hour.

speed = 80 miles/hour

Both the method of computation and the fact that I am finding speed at all assumes that I made the trip in uniform motion. But that wasn't the way it really happened. There was a lot of stopping and starting until I got to the freeway and then I let my '36 Chevy coupe with a punched out 442 have its head. I stopped at Rafael's Silver Cloud for a coke and then buzzed on in to Santa Fe. Nevertheless, I made the trip in exactly three quarters of an hour. The laws of physics hold in the Ideal World and given the route from my house to the Santa Fe city line there is no way I could have kept the speedometer on 80 the entire trip, even if I hit all the lights. Some ideal mass-less car in a 'gedanken experiment' could do it maybe.

Even though the motion of my car was not uniform and so can't, at the moment, be described by speed, the 80 miles/hour does give some idea of what the motion was like and it is called the average speed. **The speed that the object would need if it were to travel the same distance in the same time as the actual, non-uniform trip, is the average speed.** The average speed describes my trip of 60 miles in 45 minutes if the motion had been uniform. Any trip taken by an object has a distance traveled and a time of travel. The quotient of these two is how I compute the average speed.

There is a stretch of the freeway that has markers every mile for ten miles and I set my clock and odometer to zero as I passed the first one. I had to slow a little for some traffic but I thought that my average speed over this stretch was pretty close to what this puppy could do, ten miles in four minutes for an average speed of 150 miles/hour. Not bad. I might mention that there are no speed limits in the Ideal World. And no accidents except over in statistics where it can be a real blood bath.

The point is that I can find average speeds over parts of the trip. Between any two times the object travels some distance. The average speed is the speed that the object would need if it traveled the same distance in the same time in uniform motion. The distance divided by the difference in the times is the way I compute the average speed over that interval of time.

The average speed of an object is a global description of the motion over the interval of time as opposed to an instant by instant description. Averaging 50 miles/hour on the trip to Santa Fe does not mean that I won't get speeding tickets, because I could go 108 miles/ hour for half an hour and 24 miles/hour for 15 minutes and average 50 miles/hour.

The fact that I can drive with an average speed of 50 miles/hour means that I can drive the 2000 miles from Los Angeles to St. Louis in five, eight hour days. It means that if I drive for 14 hours, I will have covered 700 miles. These are more practical uses of average speed.

‘Average speed’ is defined in terms of the previously defined ‘speed’; the average speed of a trip is the speed of an equivalent trip in uniform motion.. This is typical of the way things grow in the Ideal World.

In particular, it is the way numbers arise in the Ideal World.

The counting or natural numbers are the seed. The integers are defined in terms of the natural numbers, the rational numbers are defined in terms of the integers, the real numbers are defined in terms of the rational numbers and the complex numbers are defined in terms of the real numbers. This is enough numbers for most reasonable people.

Each step is a special case of the next step. A natural number in an integer, an integer is a rational number, a rational number is a real number, and a real number is a complex number.

LECTURE 4-8

My school was a great success. I am financially well off and people seek my company for intelligent and witty conversation. I am happily married; 12 children of course...

It is my avowed purpose to express the properties of motion in terms of the function that models position and I am ready to get with the program.

I have an object in motion. It could be in uniform motion, it could be a falling rock, or it could be my car on its way to Santa Fe. It could be any object in motion.

I am going to model the distance traveled by the object by a function that I am going to call, s. The motion is going to take place during the interval of time, [0,T], so that the domain of s is that interval. I am also going to let s be the dependent variable and let, t, be the independent variable. The rule of s is given by

s = s(t) = distance the object travels between time equal to zero and time equal to t.

If the motion is uniform with speed σ, then $s(t) = \sigma \times t$. If the object is a falling rock, $s(t) = 16\,t^2$. If the object is my car, there is no algebraic expression of the rule and

s(t) = distance the object travels between time equal to zero and time equal to t,

is about as good as I can do. So whether it is one of these three motions or some other, I will call the function that models distance traveled, s.

The total time of the trip is T - 0 and the total distance traveled is s(T) - s(0). It is true that s(0) = 0 and I am often asked why I leave the zeros in the two expressions. I do it because I am always looking for patterns and I can't see them if I leave out terms. Both of those zeros represent something. In 'T-0', the zero represents the time that I start looking at the object, s(0) represents how far the object has traveled since t = 0, which is not at all.

average speed of the trip = s(T) - s(0) / T - 0.

The average speed over a part of a trip, say from t =2 to t = 6, can also be expressed in terms of functions.

[The distance traveled when t = 2] subtracted from [the distance traveled when t = 6]

= s(6) - s(2)

= [distance traveled between t = 6 and t = 2],

and

[the time elapsed between t = 2 and t = 6] is (6 - 2) = 4.

The average speed between t =2 and t = 6 is [s(6) - s(2)] / 4.

If a and b are two intermediate values of time, $0 < a < b < T$, then the average speed of the object over the interval [a,b] = s(b) - s(a) / b - a

The language is so Real Worldly that I can forget that I am in the Ideal World. The function s is a 'rule and a set of numbers'. It doesn't know what distances and times are, so it doesn't see distances and times changing as the object moves. The function sees changing numbers. If s(6) is different than s(2), then I think that the function has changed as t has changed from 2 to 6. I guess it is really the output of the function's rule that has changed, but I am going to say that the function has changed anyway. My mind continues to identify the function more with the rule and less with the domain.

The term s(6) - s(2) is called a change in the function, as is s(b) - s(a). The term 6 - 2 is a change in the independent variable, as is b - a.

average rate of change of the function s over the interval [a , b]. = **s(b) - s(a) / b - a .**

When I read 'rate of change' I am thinking 'ratio of changes'.

As often as not, I am interested in the change of a function more than the actual values the function takes. I may care more about s(6) - s(2) than I do either s(2) or s(6) and I don't necessarily have to know s(2) and s(6) in order to find s(6) - s(2). Many meters, for example voltmeters, read differences. If I change the number of square feet in my house, I am interested in the change in wall and ceiling area, because that's what I have to paint, and the change in volume because that is what I must add to my air-conditioning.

Change is another concept that is so important that it needs some notation and Δ is the generally accepted symbol that denotes change.

Δs stands for a change in the function s, s(b) - s(a), or a change in the dependent variable if it has the same name as the function. If I am thinking about motion, Δs stands for a change in the distance traveled.

Δt stands for a change in the independent variable, b - a, or a change in time if I am motion conscious.

I can express the average rate of change of a function as $\Delta s / \Delta t$.

$$\Delta s / \Delta t = s(b) - s(a) / b - a.$$

The notation Δs, Δt , and $\Delta s / \Delta t$ do not explicitly give the a and b needed to compute the Δs and Δt so the delta notation is more a convenient way to say what is going on than a way to express numbers.

No matter what the motion of an object is, there is a function that models distance traveled as a function of time. What about the other way? Does every Ideal World function model the position of some object in motion? I don't think so. The idea of 'Ideal World function' is so incredibly general that it would be very surprising if every function modeled the position of an object in motion. For example, the function, f, whose domain is the interval [0,10] and whose rule is given by

$$f(x) = 0 \quad 0 \le x \le 5$$
$$f(x) = 1 \quad 5 < x \le 10,$$

does not model Real World position. The object would not move at all for the first five seconds and then it instantaneously moves one foot and stays there for the next five seconds. This contradicts the 'principle of continuity' because the object has skipped all the points whose distance from the start position is a number between zero and one. It has jumped from zero to one and Real World objects can't jump.

There is one last item to take care of before I move on and that is consistency. The average speed of an object in uniform motion should be the same as its speed. Since I have an algebraic expression for the rule, I should be able to compute the average speed.

The position of an object in uniform motion with speed σ during the interval of time, [0,T], is modeled by the function, s, whose domain is [0,T] and whose rule is given by

$$s = s(t) = \sigma \times t.$$

The average speed of the object between t=2 and t = 6 is

$$\begin{aligned} s(6) - s(2)/ (6 - 2) &= (\sigma \times 6 - \sigma \times 2) / 4 \\ &= \sigma (6 - 2) / 4 \\ &= \sigma. \end{aligned}$$

The average speed of the object between t = a and t = b is

$$\begin{aligned} s(b) - s(a) / (b-a) &= (\sigma \times b - \sigma \times a) / (b-a) \\ &= \sigma (b-a) / (b-a) \\ &= \sigma. \end{aligned}$$

I am really glad that worked out. I would not have liked to start all over again. Evidently Kepler worked for five years trying to show that the orbit of the earth was a circle, which it very nearly is. He finally realized that no matter how much he wanted it, it wasn't going to be, and he started all over. I thought it took a lot of courage to dump five years of work. One of my favorite poets, Mick Jagger, has written, "You can't always get what you want, but if you try sometime, you just might find, you get what you need." How true, how true.

When I have to abandon a project because I realize that what I'm trying to do isn't possible, I think of Kepler.

I recall watching a nature program on television about the female tarantula wasp. I first saw her on her way to find a tarantula and instead of a tarantula, she ran into an anthill.

The ants were all over her and I thought that the program was starting with an episode showing how chancy the life of the tarantula wasp was. But she fought them off and escaped. Over a slight rise there was a tarantula which she immediately attacked. The ensuing fight was brutal. Time after time it seemed the tarantula had won and the wasp was motionless on the ground, but she struggled back to her feet and was after the huge spider again. And she won! She won! The tarantula was motionless at her feet.

Had I been the wasp I think that I would have buried the beast on the spot, but she laboriously dragged it to some apparently better place, fifteen feet away. She finished the job by burying the spider and laying the <u>one</u> egg she had done all this for, in it. When I feel daunted by a task ahead, I think of the female tarantula wasp.

Further Considerations of Chapter 4

I want to say a few words about the formula that relates the speed, v, the time, t, and the distance, s, of an object in uniform motion.

$$s = v \times t.$$

There are three problems inherent in this formula, as there are in any formula with three unspecified letters.

1. Given speed and time, find distance. This is the most direct way to use the formula. For example, if I traveled at 60 mph for two and a half hours, how far did I go?

2. Given time and distance, find speed. For example, if I traveled 45 miles in 15 minutes, how fast was I going? This problem leads to a new formula that expresses speed in terms of distance and time.

3. Given distance and speed, find time. For example, if the baby is due in half an hour and the cab, whose top speed is 90 mph, is 45 miles from the hospital, will the baby be born in the hospital or the cab? If this is considered an Ideal World problem, a dead heat means the baby is born in the hospital, in the Real World it means the cab. The whole idea that the arrival of the baby can be predicted exactly smacks of the Ideal World. Here again, one is led to a new formula that expresses time in terms of speed and distance.

Real World formulas can give rise to Ideal World rules, which give rise to Real World formulas.

If there is a current, I, in amperes, flowing through a resistor of R ohms, then the voltage drop across the resistor in volts, V, is given by $V = IR$.

This is another formula that involves three variables and so gives rise to three basic problems.

1. What is the voltage drop across a 10 ohm resistor if the current is 5 amperes?

2. What resistor will give a voltage drop of 12 volts if the current is 2 amperes?

3. What current is required to give a voltage drop of 24 volts across a 6 ohm resistor?

Another example of this is the formula that relates the pressure, P, at a depth D in a liquid whose density is ρ, $P = \rho D$. If we are working in the English system, the units of depth are feet, the units of pressure are pounds per square foot, and the units of density are pounds per cubic foot.

If I consider that ohms are really volts per ampere, then all of these formulas have a 'something per something' in them. Speed is distance per time and density is pounds per cubic foot. I tend to think of all of them as densities so that speed is the density of distance in time, for example.

Units behave in much the same way that numbers do. If I travel for ten seconds at 12 feet per second then the distance covered is

12 feet/second x 10 seconds = 120 feet.

Notice that the seconds cancel to give feet. This is the way units always work.

Why do I believe?

Why do I believe 2 + 3 = 5?
Why do I believe 200 + 300 = 500?
Why do I believe the area of a rectangle is length times width?
Why do I believe the area of a circle is pi times the square of the radius?

I once began a project to catalog the things I believed. I first realized that I didn't just believe or not believe, there were gradations of belief. I really believed some things, like the sun comes up in the morning. Well, I guess what I really believe is that the Earth turns on its axis and that the sun exists as a huge mass of burning hydrogen that seems to be fixed in space relative to the Earth. Or something like that. I believe the Earth is round because I have seen pictures of the Earth taken from the moon and it looks round. Those pictures are certainly good evidence for the 'roundness of the world' and I personally doubt the stories that the lunar trip was a hoax. And I can't just brush off the lunar eclipse. I'm pretty sure the world is round.

I believe that a water molecule has two hydrogen atoms and one oxygen atom on the basis of high school chemistry. I am fairly certain that the atoms involved consist of electrons, protons, and neutrons. I have very little idea of what electrons, protons, and neutrons are made of. My belief level is high at the atom level but since I don't really know what atoms are, I am not sure what it is I believe.

I once realized that my belief of what life in towns in the western United States was like in the 1860's to the 1890's was based on based on Hollywood's concept of history as portrayed in countless movies about the west.

There are many things I believe because somebody told me so or I read it somewhere. I was surprised at how few of my beliefs were based on my own investigations and observations.

I think that it is important to look into belief when studying mathematics because mathematics needs belief if it is to be used effectively and creatively. I think a person needs confidence in their knowledge of mathematics if they are going to use it successfully and to have that confidence a person should know why they believe the mathematics they think they know.

Sorting through one's belief system and trying to figure out why it exists is a non-trivial task.