CHAPTER 3

A function is a function is a function...

LECTURE 3-1

My earliest memory was the belief that an exactly repeated action would yield an exactly repeated result...

I'm going to measure the time it takes an object to fall some distance with a fairly straight forward device. A relay starts the clock when the object is released and another stops the clock when it has fallen the desired distance. The smallest interval of time that the clock can measure is 0.0001 second, but there is some error in the relays, some error in measuring the distance the object falls, and some error from general wear and tear on the apparatus. If the object has much weight at all, it's going to hit the bottom with considerable impact and this must be accounted for in the design. I am going to drop the object from a height of one foot to a height of sixteen feet in one foot intervals.

Even though the clock is good to 0.0001 seconds, I will round the numbers off to three decimal places. Here is a tabulation of my results.

height in feet	time in seconds	three places
1	0.2499	0.250
2	0.3537	0.354
3	0.4332	0.433
4	0.5002	0.500
5	0.5589	0.559
6	0.6125	0.612
7	0.6612	0.661
8	0.7070	0.707
9	0.7501	0.750
10	0.7907	0.791
11	0.8290	0.829
12	0.8662	0.866
13	0.9016	0.902
14	0.9352	0.935
15	0.9680	0.968
16	0.9998	1.000

Most of the numbers that appear as values of time don't do much for me right at the moment, but there are four of them that do catch my eye.

height in feet	time in seconds	three places
1	0.2499	0.250
4	0.5002	0.500
9	0.7501	0.750
16	0.9998	1.000

Evidently, the object falls one foot in the first quarter second, three feet in the next quarter second, five feet in the next, and seven feet in the final quarter second. It confirms my visual observation that the rock fell further in each succeeding interval of time, but much more, it also says that this happens in an orderly way. One, three, five, seven; now, that is orderly. *Vita dulcis, Deus bonus.*

There is a kind of mathematics I call 'kick the can' mathematics. I come from a less litter conscious age and if I saw a can in my path as I walked home from school, I kicked it. And even though social pressures have changed my habit into picking the can up and recycling it, my first impulse is still to kick it.

Whenever I see a bunch of mathematical expressions with a common factor, I factor it out. It may do me no more good than kicking the can but I do it anyway. And sometimes it does do me some good. If I factor the 0.25 out of the four values of time, I get

height in feet	time in seconds
1	0.250 = 0.25 x 1.00
4	0.500 = 0.25 x 2.00
9	0.750 = 0.25 x 3.00
16	1.000 = 0.25 x 4.00

Now, I need some relation between the heights, 1, 4, 9, and 16 and the numbers 1, 2, 3 and 4. Well, 1, 2, 3 and 4 are the square roots of the heights. For these four values of height, anyway, I have a formula that expresses the time of descent in terms of height.

$$\text{time of descent} = 0.25\sqrt{height}\ .$$

In order to avoid writing 'time of descent' and 'height' a lot, I am going to abbreviate 'time of descent' by 't' and 'height' by 's'.

I now have the rather remarkable formula

$$t = 0.25\sqrt{s}$$

or using fractions, which, for no particular reason, I personally prefer,

$$t = 1/4\ \sqrt{s}\ .$$

I don't want to count my chickens before they're hatched. This formula works in the case of four of the heights but there are twelve more. Could it happen that the formula would not work for these twelve? Not to worry. As a matter of fact, if something looks good in a few cases, it will probably look good in all cases.

This is a general philosophy of mine. I know that there are cases where a formula will work ninety-nine times and not the hundredth, and although I don't recall running into one myself, I know of people who have. They have had to struggle to get a formula that would work that hundredth time. Still, my situation is not very complex and I have no trouble believing that the four values of height I considered are representative.
I check the other cases, and just as I anticipated, the formula works for all sixteen cases, seventeen if I count the case where the height and the time are both zero. Since it works

for these seventeen values, I am going to make the big jump and believe that it works for all values of height between zero and sixteen. I believe that it works for a lot of heights greater than sixteen, but I don't know about heights like a hundred miles.

This is a Real World formula. The $\sqrt{s}$ makes it look a little 'Ideal Worldly', but in this formula, $\sqrt{s}$ is the number I get on my calculator, which is very Real World.

Now I go out of the Real World laboratory and into the Ideal World laboratory. Here the time that it takes to fall 7 feet is $\sqrt{7}/4$ seconds, and the $\sqrt{7}$ is an Ideal World number. I must decide how I want to transfer the formula to the Ideal World.

LECTURE 3-2

I began golfing at the age of six and worked tirelessly to make my swing exactly the same each time. I didn't care where the ball went, only that it went the same way every time I hit it...

The question of how to put the formula in the Ideal World requires some careful thought. The formula gives the numbers but there is more to what is going on than just the numbers. There is the idea that given a height there is a unique time; the 'principle of uniqueness' is here. The formula doesn't work for all values of height, in particular, it doesn't work for six feet if I drop the object five feet from the floor. Somehow the applicability of the formula has to be taken into account. The motion that the formula describes satisfies the 'principle of continuity' and this should be in it somewhere.

What the formula actually <u>does</u> is associate every Real World height between zero and sixteen feet with a unique Real World time<u>. In the Ideal World I will model the formula with a **rule** that associates every Ideal World height between zero and sixteen feet with a unique Ideal World time</u>. The rule uses Ideal World numbers in the formula to make the association.

Since the rule and the formula are represented by the same algebraic symbols, there is nothing about the written expression that tells which world it is in. I express that distinction by calling it the rule, $t = 1/4\sqrt{s}$, or the formula, $t = 1/4\sqrt{s}$.

I am making a distinction between rules and formulas here that is not often made, if ever, and I do it because I want to keep the Real World and the Ideal World separate in my thinking. In practice they are used interchangeably and context makes clear which actually applies.

This being the case, it may seem artificial to separate the ideas of rule and formula. Rules, however, are not always given by formulas. It is a much more general idea. The rules I run into everyday are usually given by formulas but many others are not. For example, I can associate each positive integer, n, with the number in the n th place of the decimal expansion of $\sqrt{2}$. This is a perfectly good rule but there is no formula.

My Ideal World model for the Real World position of a falling rock is a 'set of numbers' and a rule that associates each number in that set with a unique number. The 'set of numbers' is the set of numbers that represent the Ideal World heights between zero and sixteen feet. The formula gives the numbers that represent the times that it took the rock to fall from these heights. The rule is, "apply the formula to the Ideal World numbers representing height and get the associated Ideal World time". By means of the formula, the rule associates every number representing distance with every number representing time.

Since the numbers that represent time depend, via the rule, on the numbers that represent distance, knowledge of the heights and the rule implies knowledge of the times. All that I need to describe the time to fall with respect to height is the set of heights and the rule.

The **set of numbers and the rule that is defined upon them** is called a **function.** The set of numbers is called the **domain** of the function. I will say that the **function is defined on the domain.** The set of numbers that result from applying the rule to the domain is called the **range**. Since the rule assigns a single number to each element of the domain, as opposed to, say, four numbers, I call both the rule and the function, **single valued.**

The purpose of the function is to capture the essence of the position. I say that I have that essence if I know how long it takes the rock to fall every distance or, in terms of points, how long it takes the rock to get to every point on the path. This is exactly the information that the function gives me. It supplies the set of heights and the rule that turns the heights into times. This is not the complete description of motion, but I have a good start.

I express the fact that the function relates distance to time by saying that the **'function models time as a function of distance'.**

Function is an entirely general idea. Any set of numbers and any single valued rule defined on it, is a function. Functions are Ideal World objects and while most of them do not model anything in the Real World, they can still be used for fun and profit.

I said that a function is a set of numbers and a rule defined on that set of numbers but I sometimes call these numbers 'heights' or 'times' or some other physical quantity. A set of numbers is a set of numbers. Whether they represent height, voltage across a resistor or the angular momentum of a dying star is all in my head. The numbers don't know how they are being used; I know. I call them 'heights' and 'times' so that I can keep track of what's going on.

For a given amount of power between zero and sixteen watts, there is a unique value of electric current in amperes, that will dissipate that much power in a sixteen ohm resistor. The formula for this value of current is given by

$$\text{current} = \sqrt{power}\ / 4.$$

If I call current, i, and power, p, then the formula becomes

$$i = 1/4\sqrt{p}$$

I can model the power dissipation in a 16 ohm resistor by the function whose domain is the set of numbers between 0 and 16, inclusive, and whose rule is "take the square root of a number and divide it by 4".

The function that models power dissipation in a 16 ohm resistor is the same one that models the position of the rock, namely, the function whose domain is the set of numbers between zero and sixteen and whose rule is to divide the square roots of numbers in the domain by four. I don't really have to understand anything about electricity to understand that the same function is modeling two very different physical phenomena.

If the numbers in the domain of a function represent feet, then I will call those numbers feet and think on them as Ideal World feet. If they represent watts of electrical power, then I will call those numbers watts and think of them as Ideal World watts being dissipated by an Ideal World resistor that is exactly 16 ohms.

The power in the concept of function is that it depends on the form of the relation between physical quantities, and not on what the physical quantities actually are. The domain of a function is a set of numbers and the rule associates a number with a number. The function neither knows nor cares what these numbers represent.

If I want to emphasize this independence from the Real World, I can write the rule and domain as

$$y = \sqrt{x} / 4, \qquad 0 \le x \le 16$$

and not talk at all about feet or watts or units in general. But if I am interested in the rock I just dropped from a sixteen foot shed, I am going to write the rule and domain as,

$$t = \sqrt{s} / 4 \qquad 0 \le s \le 16,$$

and talk about letting *s* equal 4 feet and getting *t* equal to 0.50 seconds.

LECTURE 3-3

I practiced until each swing seemed an exact replica of the others. Examining video tape revealed no differences between one swing and another. The flight of the ball was something else again...

It can be argued that the concept of function is the central concept of mathematics and that mathematics is the study of functions. I think I might argue that point of view and that being the case, I think I'll look at this idea just a little longer.

First, I'm going to develop some notation and language. A function consists of a set and a rule and it is fairly cumbersome to keep referring to it by its constituents. Usually I give it a name which is a single letter, like f, *g*, or h. Sometimes I will use a letter that has a special significance in the model, like the first letter of the physical quantity that is represented by the numbers in the range of the function.

I am going to call the function that models the falling object, f. The domain of f is the set of numbers between 0 and 16 inclusive. The rule of f is to take a number from the domain, take its square root and divide the result by four.

If I let s stand for any number in the domain, then f(s) stands for the number that the rule associates with s. I will sometimes say that **f(s) is the image of s under f.** For example, if s = 4, then $f(s) = f(4) = \sqrt{4} / 4 = 2 / 4 = 1 / 2$ and 1/2 is the image of 4 under f. I will also say that the rule of f **returns** the number 1/2 for the value 4.

This notation brooks no exceptions. Whatever appears in the parentheses gets the rule applied to it. For example, $f(sam) = \sqrt{sam} / 4$. I will worry about what 'sam' might mean later, but for right now, I'll take its square root and divide by 4.

While the verbal statement of the rule works and is often the only way to express the rule, it is generally agreed that algebraic notation works best if it is possible. The rule of the function, f, can be written algebraically as

$$f(s) = \sqrt{s}/4.$$

Since f(s) gives the 'time' it takes an object to fall a distance s, I am going to let t stand for the numbers in the range, express the rule as

$$t = f(s) = \sqrt{s} / 4$$

and say that **t is a function of s**. I verbalize , t = f(s), by saying that 't' or 'f(s)' is 'the number returned by the rule of the function for the value, s. Or I might say that the function, f, returns the value 't' for the value , 's'. Or I might say something rather like these examples but not exactly the same.

One of the things that I learned early in my career was that there were a lot of different ways to say exactly the same thing. I first thought that each of these verbalizations was different in some subtle way and I searched diligently for these differences. I finally decided that there often was no difference. Of course, I get bitten every once in a while when there is a difference.

If I am thinking about the model, I would say time is a function of height. I might also say that f is the function defined on the set [0,16] given by

$$t = f(s) = \sqrt{s} / 4 .$$

In the expression, t = f(s), the ‘t’ and the ‘s’ stand for numbers that vary over sets. The ‘s’ varies over the domain and the ‘t’ varies over the range. Not surprisingly, these symbols are called **variables**. Whatever symbol is used for the domain variable is called the **independent variable**, and whatever symbol is used for the range variable is called the **dependent variable**. The idea is that I can pick the value of the domain variable freely, independent of any restriction except that it be in the domain, while the value of the range variable depends on the value of the domain variable and the rule.

There are those who say that a variable is a symbol that can be replaced by a number from a set of numbers. I guess that is OK. It’s certainly not immoral.

In the case of the falling rock, the rule is given by

$$t = f(s) = \sqrt{s} / 4 .$$

I use s for the independent variable and t for the dependent variable.

There are times when the physical quantity represented by the numbers in the range actually depends on the physical quantity represented by the numbers in the domain. If the temperature of a confined gas goes up, the molecules move faster making the pressure inside the container go up. The pressure seems to depend on the temperature. If I were going to model this with a function, I would make the numbers that represent the available temperatures the domain and then try to find a rule that associates the numbers in the domain with the numbers that represent pressures.

But very often, time is the physical quantity represented by the numbers in the domain. I see no causal relationship between time and another physical quantity in the same way that I see a causal relationship between temperature and pressure. I see the meaning of independent and dependent as residing more in the dependence of the number in the range on the independently chosen number in the domain and the rule, and less in the relation between physical quantities.

I will often say that I going to make some physical quantity like time or temperature the independent variable. What I mean is that I going to let the numbers in the domain represent time or temperature. I express the dependence of pressure on temperature by saying that **pressure is a function of temperature.** In the case of the rock I have modeled time as a function of height.

Let F be any function in the Ideal World. By this I mean that the domain of F can be any set of numbers and the rule of F can be any single valued rule defined on that set. I'm talking a high degree of generality. This function does not necessarily model anything in the Real World and in this case I almost always let the independent variable be x and the dependent variable be y. I say that y is a function of x and write y = F(x).

Choosing x to be the independent variable is a long and honored tradition.

I talk about letting s be a number in the domain of the function and letting t be a number in the range. That's what I say, but what I'm thinking is, "...let s be the distance that the object falls..." and "...let t be the time it took to do it...". A person's mind is a hard thing to control.

LECTURE 3-4

I decided that the problem was atmospheric conditions over which I had no control. I took up bowling...

When I model a physical phenomena using a function, the domain is usually self evident because the physical thing is sitting right in front of me in one sense or another. I know that the heights I am considering are between zero and sixteen feet and the domain is not a problem. The rule is the problem.

In the case of the falling rock, my laboratory data led me to a rule that I could express both verbally and algebraically, but I don't have to look very far to find examples where that doesn't happen.

As I watch the leaves fall in Autumn, I believe that at every Real World instant of time each leaf is at a unique Real World distance from the ground.

In the Ideal World I can model the position of each leaf by a function whose domain is the set of Ideal World numbers that represents the Ideal World interval of time that it takes the leaf to fall to the ground, and whose rule associates each instant of time in the interval with a unique Ideal World number that represents the unique Ideal World distance between the leaf and the ground.

This is a very complex sentence. Maybe it looks better like this.

In the Ideal World,
I can model by a function
the position of each leaf whose domain is

the Ideal World set of numbers that
represents
the Ideal World interval of time

that it takes the leaf to fall
to the ground

and

whose rule each instant of time
associates in the interval with

a unique Ideal World number that

represents

the unique Ideal World distance

between
the leaf and the ground
at that instant of time.

Or this:
In the Ideal World I can model the position of each leaf
by a
function

whose
domain
is
the set of Ideal World numbers that
represents
Ideal World times

and
whose
rule
associates
each instant of time in the interval
with
a unique Ideal World number
that represents
the unique Ideal World distance
between the leaf and the ground.
at that time.

I could be a little more economical with words and say that I am going to model the position of the leaf by a function of time, g.

Unfortunately, I have no algebraic way of expressing the rules of these functions. Each falling leaf is an irreproducible experiment and I have no way to gather data that I can examine for patterns. If I had set up a camera and filmed the fall of a particular leaf, I could get data but it would show no pattern. There is no algebraic way to express the association of time and the leaf's distance from the ground. I guess I don't really see why it's unfortunate that these rules can't be written.

But even if the rule is unknowable and the particular numbers that represent the distances are not knowable, the rule is perfectly well defined and exists. Consequently the function exists. At every time the leaf has to be somewhere and somewhere has to be a fixed, singular distance from the ground. Just because I don't know something doesn't mean that it doesn't exist. I hope that's true.

The importance of function, however, lies not so much in having an algebraic formula for the rule, but in that everything I know or want to know about the model can be expressed in terms of a function.

LECTURE 3-5

I had even less success at bowling although I could approximate my goal if aimed at the gutter. Throwing the perfect gutter ball was not the skill I sought...

I would eventually like to express all the properties of the motion of an object in terms of the functions that model the motion. I'm going to start with the concepts of uniqueness and continuity.

I will use the ‘leaf’ function, g, to make the ideas concrete. In this case $y = g(t)$ = height of the leaf above the ground at time, t.

The Real World ‘principle of uniqueness’, that for every instant of time, the leaf is a unique height from the ground, is expressed in the idea that the rule of the function is single valued; that for every value, t, in the domain of the function, the rule gives a unique value $y = g(t)$.

The Real World ‘principle of continuity’ says that if the leaf is at two different heights from the ground at two different times, then the leaf will be at every height between these two heights at some time between the two given times. The leaf will be at every intermediate height and some intermediate time.

The Ideal World ‘principle of continuity’ says that when the rule of the function returns two different numbers in the range for two different numbers in the domain, then the rule will return every number between the two range numbers for at least one number between the two numbers in the domain.

This statement is correct but hard to read. The two different numbers in the range and the two different numbers in the domain are awkward to refer to, as are the numbers between them. I think that there comes a time when statements are more readable when written using mathematical notation, and that time may have come. No matter which way I write it, pretty much the same thing goes on in my head. Here is the ‘principle of continuity’ as a mix of Real World and Ideal World.

If the leaf is at two different heights, $g(2) > g(6)$, at two different times, $2 < 6$, then for any intermediate height, y, $g(2) > y > g(6)$, there is an intermediate time, t, $2 < t < 6$, when the leaf is at the height, y, and $g(t) = y$.

Removing the leaf and looking only at the function, I can say:

If the rule returns two different numbers in the range, say $g(2) \geq g(6)$, at two different numbers in the domain, $2 < 6$, then for any number, y, between the two range numbers, $g(2) < y < g(6)$, there is some number, t, between the two domain numbers, $2 < t < 6$, where the rule returns the given number in the range, $g(t) = y$.

And now, I take away as much language as I can.

For any number, y, that satisfies $g(2) < y < g(6)$, there is a number, t, $2 < t < 6$, such that $g(t) = y$.

This last statement expresses the ‘principle of continuity’ entirely in terms of function and number; all trace of the Real World that I am modeling has been removed.

I again need some notation. The set of all numbers between two numbers keeps coming up, as in the set of numbers between 2 and 6 of the previous example. Such sets are called **intervals** and there are four of them.

$$[a , b] = \{x| a \le x \le b\}$$

$$[a , b) = \{x| a \le x < b\}$$

$$(a , b] = \{x| a < x \le b\}$$

$$(a , b) = \{x| a < x < b\}.$$

If a point, z, is contained in the interval, (a, b) = {x| a < x < b}, then (a,b) is called a neighborhood of z. Only intervals of the form (a,b) can be neighborhoods and (a,b) is a neighborhood of any point it contains. This terminology is descriptive. All the points in (a,b) are neighbors, and (a,b) is their neighborhood.

Using the interval notation in the statement of continuity:

If the domain of the function contains an interval, [a,b], and if g(a) < y < g(b) or g(b) < y < g(a), then there is a number, t, in the interval, (a,b), such that g(t) = y.

I see no vestige of the Real World in this statement of the ‘principle of continuity’ but apropos of the old saying, “Don't think of an elephant.”, I can't help thinking the Real World into it. The g(a) and the g(b) are two different heights, y is an intermediate height, the a, b, and t are times.

There are functions that do not have this property but every function that models a continuous physical process does have it. I am going to single out the functions that model continuous physical processes by calling them **continuous**. I don't want to commit myself to this being all of the functions that I want to call continuous, but, whatever I finally decide a continuous function should be, functions that model continuous physics should number among them.

There are other qualities of the motion that I can express in terms of function.

If it is a fairly calm day, the leaf always gets closer to the ground. This is reflected in the function by saying that if a and b are in the domain and a < b then g(a) > g(b). I would say that the function is **decreasing**.

If the leaf were rising, then there would be values in the domain, a and b, where a < b and g(a) < g(b). A function is **increasing** on an interval if for any two values a and b in the interval, a < b, I have g(a) < g(b).

On a little breezier day, the leaf may actually rise for brief periods of time before the inevitable end of its journey on the well-kept lawn below. Since the leaf goes up and then back down, there are at least two different values of time when the leaf is the same distance from the ground.

In terms of the function, I would say that there are distinct numbers a and b in the domain where g(a) = g(b).

If the leaf goes up and then down, there must be three values of time, $a < c < b$, where $g(a) = g(b) < g(c)$. That is to say, there are two values of time, a and b, when the leaf is the same height from the ground, and a time in between when it is higher. There must also be some greatest height of the leaf above the ground in this little maneuver and so there must be a time, t_0, $a < t_0 < b$, when this happens. In terms of functions, there is a number,
t_0, $a < t_0 < b$, where $g(t_0) > g(t)$ for any t in (a,b) that is not equal to t_0.

If this happens, I say that the function g has a **local maximum** at t_0 and the local maximum is $g(t_0)$. I use the word 'local' because $g(t_0)$ is a big fish only in the neighborhood, (a,b). When the leaf is close to where it broke from the tree, it will be higher than $g(t_0)$. Or I could say that for values of t close to zero, $g(t_0) < g(t)$. The number $g(0)$, the height at the moment the leaf leaves the tree, is an **absolute maximum.** Unless, of course, it is a really windy day.

I will give a version of the local maximum that is pretty formal. The symbol D will stand for a set of numbers.

A function g has a domain D. It has a **local maximum** at $t_0 \in D$ if there is a neighborhood of t_0, (a,b), $a < t_0 < b$, where (a,b) is contained in D and such that $g(t) < g(t_0)$ if $t \in (a,b)$ and $t \neq t_0$. $t \in (a,b)$ means that t is in the interval (a,b). This just says that $g(t_0)$ is the biggest that g(t) gets if t is in the interval (a,b).

There are gains and losses in understanding no matter how the local maximum is described. The Ideal World, function approach to the description of a local maximum is brief and precise but on the arid side. On the other hand, the words that give the same meaning as the symbols are often hard to find and sound clumsy and stilted when I do. What I <u>do</u> is write in a fairly symbolic, Ideal World way, but as I read, my mind translates it into terms I can relate to more intuitively.

The description of motion will be done in terms of the function that models position. Anything that I want to say about the physical aspects of this motion, I can say using this function or with functions I derive from it. In practice, however, I have to amplify with Real World words what I say using functions.

LECTURE 3-6

While walking in Central Park I over heard a jogger tell her companion that mathematics was the most exact science. I should have been looking at mental, not physical activities...

In my model of the falling rock, the distance the rock fell was the independent variable and time was the dependent variable. In the model of the falling leaf, time was the independent variable and height from the ground was the dependent variable. My intuition tells me that for each time there is a unique distance that the rock falls, so it seems that I should be able to model the falling rock using time as the independent variable.

There are two approaches to this problem. One is to work in the Ideal World and use the methods of mathematics, in particular algebra, to find the rule that associates time with the distance the object falls. The other is to go into the laboratory and ferret out the rule from data. What I am going to do is find the rule in the Ideal World and then verify that it is correct in the Real World by experiment.

As before I'm going to let t stand for time and s stand for the distance the object falls in that time. If h is the name of the function whose rule associates time with distance, then I can write the rule

$$s = h(t).$$

I would like to find some algebraic way of expressing this rule. Well, I have to start with what I know and that is

$$t = \sqrt{s} / 4 .$$

I want to turn this into s = 'something' so I must liberate the s which is tied up inside a square root and a division by 4. I'll get rid of the 4 first by multiplying both sides of the equation by 4 to get

$$4t = \sqrt{s} .$$

Then I'll eliminate the square root by squaring both sides of the equation and my emancipated s can be written

$$s = 16\, t^2 .$$

Congratulations! It's a rule.

I let h be the function whose domain is [0,1] and whose rule is given by $s = h(t) = 16\, t^2$, where t is the time and s represents the distance the object the object falls in that time.

The laboratory apparatus consists of a long, vertical metal rod, a steel ball which is dropped right next to the rod, and a power source that applies a high voltage to the rod every tenth of a second. A long piece of tissue paper is between the rod and the ball and each time the voltage is applied there is a electric arc from the rod to the ball which burns a little hole in the tissue paper. The position of the ball on its downward journey is marked by a hole burned in the tissue paper every 0.1 sec.

The first pulse of voltage is at the instant the object is dropped and the distance from the first to the, say, third hole is how far the object fell in 0.2 seconds. Generally, the distance from the first hole to the n th hole is how far the object fell in (n-1) tenths of a second. The data looks like this after I have rounded off to two places:

t = time	s = distance
0.00	0.00
0.10	0.16
0.20	0.64
0.30	1.44
0.40	2.56
0.50	4.00
0.60	5.76
0.70	7.84
0.80	10.24
0.90	12.96
1.00	16.00

The way I proceed from here is personal and I suppose that ten different people would do it ten different ways.

I prefer to work with integers so I am going to start by factoring out 0.01 from each distance. I also see that 16 is a factor of the first two numbers of the resulting column and a little long division convinces me that it is a factor of the rest of the numbers also. The progression looks like this:

t = time	s = distance		s	
0.00	0.00	0	0	0
0.10	0.16	16×0.01	$16 \times 1 \times .01$	$16 \text{ x } 1^2 \times .01$
0.20	0.64	64×0.01	$16 \times 4 \times .01$	$16 \text{ x } 2^2 \times .01$
0.30	1.44	144×0.01	$16 \times 9 \times .01$	$16 \text{ x } 3^2 \times .01$
0.40	2.56	256×0.01	$16 \times 16 \times .01$	$16 \text{ x } 4^2 \times .01$
0.50	4.00	400×0.01	$16 \times 25 \times .01$	$16 \text{ x } 5^2 \times .01$
0.60	5.76	576×0.01	$16 \times 36 \times .01$	$16 \text{ x } 6^2 \times .01$
0.70	7.84	784×0.01	$16 \times 49 \times .01$	$16 \text{ x } 7^2 \times .01$
0.80	10.24	1024×0.01	$16 \times 64 \times .01$	$16 \text{ x } 8^2 \times .01$
0.90	12.96	1296×0.01	$16 \times 81 \times .01$	$16 \text{ x } 9^2 \times .01$
1.00	16.00	1600×0.01	$16 \times 100 \times .01$	$16 \text{ x } 10^2 \times .01$

Each entry in the last column corresponds to a time, for example, $16 \times 7^2 \times .01$ corresponds to the time, 0.70 seconds. The number that is squared in each entry is ten times the corresponding time. In my example, the number that is squared is 7 and the time is 0.70 seconds.
I can now write a crude form of the formula and then clean it up.

$$s = 16 \times (10t)^2 \times 0.01$$

$$s \times 100 = 16 \times 100\, t^2$$

$$s = 16\, t^2 .$$

I now have a formula that relates distance to time. This formula becomes a rule in the Ideal World and I notice with some pleasure that the rule I extracted from the data is the same as the rule I obtained algebraically.

LECTURE 3-7

For the past several years I have been multiplying 8,976,643 times 11,873,356 and have yet to get the same answer twice. I must have picked the wrong two numbers...

The modern concept of function was presented by Dirichlet in 1821 and it has remained unchanged since then. There are not many things that have remained unchanged since 1821 so it must have something going for it.

As with any idea that has been around a long time, it has developed a myriad of facets and is used in every branch of mathematics, often with a name that is particular to the discipline.

Mathematics is the study of sets and the functions defined on them. This is my opinion anyway. In linear algebra the sets are called vector spaces and the functions are called linear transformations. In algebra the sets are called groups, rings and fields and the functions are called homomorphisms and isomorphisms. In analysis the sets are called topological spaces and the functions are called homeomorphisms. In probability the sets are called probability spaces and the functions are called random variables.

The domain and range of the functions I introduced to model the position of a falling object are sets of numbers. There is nothing in the idea of function that requires this restriction. The domain and range of a function called a 'linear operator' can be sets of functions. In real analysis there is a function called a 'measure' whose domain is a set of sets.

But regardless of what functions are called or what kind of elements are in the domain and range, a function is still a set and a single valued rule defined on it.

There is an infinity of functions, and it's one of the big infinities, but relatively few of them appear in modeling the physical world. There are five basic functions whose rules can be expressed algebraically and there are six ways of combining them to obtain all the rest.

I try to get acquainted with functions and make them my friends. Each has its own special character and personality and if I get to know what those are, I can understand why they behave the way they do.

I give a function numbers and it gives me back numbers. Sometimes I can see a pattern in the numbers it gives back and I can see the rule that the function is using. This is one of the ways I get to know the character of a function.

I get to know functions by seeing how their rules handle the sum of two numbers or the product of two numbers, much as I get to know people by seeing how they handle different situations.

I can draw pictures of functions and see how each of them is different. Unlike people, the character of a function is often betrayed by the way the function looks. There was a time when I looked on a page of functions pretty much as I looked on a herd of cows, but now

I see a page of familiar faces and I know how they are going to react in the situations in which they find themselves.
A fragment from The Tale of Two Cities that sticks in my mind is “the ubiquitous Jaques” and I cannot help but think, “the ubiquitous function”. As with anything so pervasive, the language around it is very rich and there are a lot of ways to say pretty much the same thing.

The following are different ways to think about the expression $t = f(s) = \sqrt{s} / 4$:

1. If I let s stand for a number in the domain and let t stand for the number in the range that the rule associates with s, then I can write the rule as

$$t = f(s) = \sqrt{s} / 4 .$$

Here, I am thinking of the function as an Ideal World object whose sole business is to associate numbers with numbers. I am thinking of s and t as numbers.

2. If I let s be the symbol that can be replaced by any of the various numbers in the domain and let t be the symbol that can be replaced by the associated number in the range, then I can write

$$t = f(s) = \sqrt{s} / 4 .$$

I am still in the Ideal World but I am now thinking of t and s as symbols that can be replaced by numbers. That is, I'm thinking of them as variables. I formally admit this in the next statement.

3. If I let s be the independent variable and t be the dependent variable, then I can write

$$t = f(s) = \sqrt{s} / 4 .$$

4. If I let s be the distance the object has fallen and let t be the time it took to get there, then I can write

$$t = f(s) = \sqrt{s} / 4 .$$

In this last rendition, I think of s and t as standing in the Ideal World, encompassing their roots in the Real World and their future as the independent and dependent variables of the function f. The symbols s and t are packed with meaning.

I don't pick any one way to think about the function and the symbols I'm using, I think about them in all the ways. I don't pick one aspect of my friends to always think about, but consider them as the total person they are.

It is true that I like some functions better than others, but I always try to remember that in the Ideal World, God loves all functions equally.

The examination of data for a pattern provides some of my rules and the manipulation of known functions in the Ideal World provides the others. In the latter case I must examine data to verify that the rule I got in the Ideal World works in the Real World. The ‘proof of the Ideal World pudding’ is in the ‘Real World eating’. I don't use a function to model a physical event until I have verified by experiment that it actually works. A consequence of this is that the rule is only as good as the measurements.

Einstein's theory of relativity became necessary when relativistic effects could be measured. When electrons could be accelerated to speeds close to that of light, the increase in mass could be measured. Clocks became so accurate that time dilation in moving reference frames could be measured. Distances could be measured well enough to detect the contraction in meter sticks. The rules of the functions Newton used to model mechanics had to be changed to account for time dilation, the increase in mass, and the shrinking meter stick.

One of the circumstances that led to the theory of relativity was the ability of Michelson and Morley to measure the speed of light accurately enough to conclude that the speed of light was the same in all reference frames. Newton didn't have this ability and so did not consider the consequences of an effect he knew nothing about.

I don't consider the consequences much myself. I use Newton's functions for the same reason Newton did, I don't travel very close to the speed of light very often, nor do the objects that I come into contact with. My clock doesn't slow down much on my drive to the supermarket. At the speeds that I travel, Newton's model works just fine.

There is a book, Mr. Tompkins in Paperback, that describes how the world would look if the speed of light were 20 miles per hour. I don't think that the contraction of a meter stick or the increase in mass would bother me as much as time dilation. There are certain constants in my life that I would just as soon stayed that way. The earth under my feet and the way time passes are two of these.

I have immense respect for Newton's work. When I read pieces of the Principia I see a man who feels he has a really neat idea and he is earnestly trying to get it across to me.

Further Consideration of Chapter 3

I want to make some fairly extensive remarks about functions that have explicit algebraic rules. There was a time when these were the only functions, and even though today the concept of function is generalized far beyond this, the functions with algebraic rules are the ones met in applications.

It is customary to group these functions according to the general form of their rules. The groups have names like ‘polynomial functions’, ‘trigonometric functions’, and ‘rational functions’ and these groups of functions are usually studied separately.

When I study the functions in a group, I am not concerned so much with the particular number the rule returns, but with the relations that the rule itself satisfies. If f is the function whose rule is $f(x) = 2x$, I am not interested in the fact that $f(2) = 4$, but in the fact that $f(x + y) = f(x) + f(y)$. If I do have the rule explicitly, I use it to find general relationships, more than compute numbers.

There are a lot of functions and at first glance the task of organizing them for study seems formidable. Fortunately, it turns out that it is not all that hard.

The way I look at it, there are five basic rules and five ways to combine them to get all the rules of all the functions that have rules. This is another of my sweeping statements that could be contested but I’ll stick with it.

I am not thinking too much about the domains of these functions. The domain of a function is usually determined by the physical situation when I’m modeling, or determined by the rule if I’m not modeling. In the later case, the domain is the <u>set of all numbers that make sense in the rule</u> and this is called the **natural domain** of the function. When I introduce these functions it will be with their natural domains.

If the domain of a function is the natural domain of the rule, then knowing the rule is the same as knowing the function. This is because the rule completely determines the natural domain of the function, and if I know the rule, I know everything. In this case the words, ‘rule’ and ‘function’ are interchangeable and I’m going to talk about a basic ‘function’ instead of a basic ‘ rule’

I’m first going to give the five basic functions and then I will talk about the five ways to combine them.

The first function is more like a class of functions. A rule in this class takes a number and raises it to a power. A function, f, is in this class if its rule looks like $f(x) = x^a$ where a is a real number.

If I am going to use this rule effectively, I have to give some thought to what 'raising a number to a power' means. This is fairly straight forward if the power is a positive integer. For example, $f(x) = x^5$, is the algebraic expression of the rule that takes a number and multiplies it times itself five times. I can multiply any number by itself five times so this rule makes sense for all numbers and its natural domain is the set of all numbers. So included in this class of functions are functions, f, whose rules are of the form $f(x) = x^n$, where n is a positive integer.

I can also use negative integers for powers, for example, $f(x) = x^{-3}$. Of course, x^{-3} does not mean to multiply the number, x, times itself a 'minus three' times, it means $1/x^3$. I am not sure how other people do it, but when I <u>see</u> x^{-3}, I <u>think</u> $1/x^3$. If the power is a negative integer, zero is not in the domain of the function because I can't divide by zero.

I consider $x^{1/3}$ to be the cube root of the number, x, and more generally, $x^{1/n}$,to be the n^{th} root of the number x. I don't think of $x^{1/3}$ as multiplying x times itself 'one third' of a time. The n^{th} roots of negative numbers are not well defined so the natural domain of functions of the form $f(x) = x^{1/n}$, where n is a positive integer, is the set of numbers that are greater than or equal to zero.

If I allow the fractional exponent to be negative, I get functions of the form, $f(x) = x^{-1/n}$, which means $f(x) = 1/x^{1/n}$. The number 0 as well as the negative numbers must be excluded from the natural domains of these functions, and the natural domain of $f(x) = 1/x^{1/n}$ is the set of all positive numbers.

I can deal with any fractional exponent as the combination of a root and a power. In particular, $x^{m/n} = (x^m)^{1/n} = (x^{1/n})^m$. <u>I can make some sense of the rule if the power is a rational number.</u>

Finding meaning in something like $x^{\sqrt{2}}$ is more of a problem. I am not taking a root and I don't see what multiplying x times itself the 'square root of two' times might mean. Well, I'm not going to say exactly what I mean by $x^{\sqrt{2}}$, or x raised to any irrational power for that matter, but I am going to use numbers raised to irrational powers as if I did know what they meant. I am going to assume that all the laws of exponents hold, regardless of the nature of the exponent. I will say $x^{\sqrt{2}} \times x^{\sqrt{3}} = x^{\sqrt{2}+\sqrt{3}}$ and not give it another thought.

Well, I guess I will give it a little thought. The transition from an expression using rational numbers to the expression using irrational numbers is vexing. On the one hand, it seems almost obvious that if a formula works for rational numbers it should work for all numbers. After all, a number is a number. Further, since rational numbers are all I have in the Real World, the question of whether the formula works for irrational numbers or not seems beside the point.

On the other hand, I am modeling in the Ideal World and solving the problem in the Ideal World. The solution is arrived at using Ideal World rules and if I am to have confidence in the solution, I must follow those rules without deviation. Is the transition from rational to irrational in line with Ideal World rules?

The success of the method depends on the Ideal World solution accurately representing the Real World. If the solution fails this test, I want to be able to blame the way I have modeled and try to change that, not worry if I have violated any Ideal World rules in the process of solving my problem. The irrational numbers are so strange and elusive that I can believe them capable of all kinds of anti-social behavior and I view this transition with some disquiet.

The actual state of affairs is comforting. In the expressions that arise in modeling the Real World, and particularly in the functions used to model the Real World, it is always OK to make the transition from rational to irrational. And that's the name of that tune.

Having said that it is OK to use irrational exponents, I still have a problem; I don't know what $x^{\sqrt{2}}$ means. Even though I am willing to use x^a as a legitimate number and use it in, say, the laws of exponents if a is irrational, I have no way to compute the number. This doesn't bother me because I am not interested in the numbers, I am interested in the laws these functions satisfy, such as $(x^a)^b = x^{ab}$.

Fortunately, there are people who know how to approximate these numbers for Real World use and they have made them available to all of us via calculators.

I am going to call a function whose rule is of the form x^r where r is a fixed real number, a **power function**. I want to emphasize that the exponent is a constant in a power function. If r is a positive integer, the natural domain of the power function is the set of all numbers. If r is a positive, non-integral number, the natural domain is the set of non-negative numbers. If r is a negative integer, then the natural domain is the set of non-zero numbers. If r is a negative, non-integral number, then the natural domain is the set of positive numbers. If r = 0, the function is a constant function and the rule returns the value 1 for all real numbers.

There was a mathematician who didn't number the pages in his book until he had formally introduced the positive integers. I don't breath air that is quite so rarefied. If I put a number in an exponent, I expect it to act like an exponent even if the particular symbol is not precisely defined. If the laws of exponents change when I use irrational numbers as exponents, then it isn't a very good law and I wouldn't be studying it. The fact that the laws are widely used and studied gives me faith that anything that walks like an exponent and talks like an exponent will act like an exponent. Eventually I would like to make a good definition for general exponents and I have faith that there is one. Formally, the laws of exponents work whether I know what the actual numbers are or not. They are Ideal World objects and they obey the laws in their particular part of the Ideal World.

I am not going to give the explicit algebraic form of the rules of the other four functions but am going to approach them somewhat differently.

In the Ideal World Scots are defined as people born in Scotland. They all have red hair, read Robert Burns and play the bagpipes. Some Danes have red hair but never read Robert Burns. No Swedes have red hair but they do read Robert Burns and play the bagpipes. I now have two ways of specifying Scots. First, I can specify Scots as people who are born in Scotland. Second, I can specify Scots as red headed people that read Robert Burns and play the bagpipes.

The first way of specifying Scots corresponds to specifying a function by explicitly giving the rule. The second way of specifying Scots corresponds to specifying a function by giving other properties of the rule which uniquely determine it. I want to ferret out what these other properties might be.

One of the things that is carefully considered when studying functions is how the rules handle the basic arithmetic operations; addition, multiplication, subtraction, and division. I want to look at f(x+y), f(xy), f(x-y), and f(x/y). These expressions are not always simple enough to use but simplicity happens often enough to make their consideration worthwhile. Further, the way that a rule deals with an arithmetic operation along with a couple of other conditions may be enough to specify the function with out giving the rule explicitly.

So instead of giving the rule of the second function explicitly, as I did in the case of the power function, I am going to give a basic relation that the rule satisfies. The relation describes how the rule of the function relates to one of the arithmetic operations, addition. This relation along with one condition can be thought of as a kind of backdoor definition of the rule, and often it is possible to find explicit evaluations of the rule for specific numbers using these relationships.

The **exponential function, exp,** satisfies the 'functional equation'

$$\exp(x + y) = (\exp x)\ (\exp y),$$

which tells how exp deals with the addition of numbers in its domain.

Since the function that is identically zero satisfies this relation and I am not very interested in that function, I am going to add the condition that exp is not the zero function. These two conditions define a family of functions and the members of this family differ from one another by what the value of exp(1) is. I am going to set **exp(1) = b** and write $\exp_b$ to show the dependence on 'b'. Popular choices for b are 2 and 10, where $\exp_2(1) = 2$ and $\exp_{10}(1) = 10$. I have my own personal favorite and I will choose it later for calculus reasons.

The functional equation can be extended to more than one summand,

$\exp_b (x + y + z) = (\exp_b x) (\exp_b y) (\exp_b z)$,
and so on.

With these two conditions I can begin to evaluate the rule for some numbers in its domain.
First, $\exp_b 0 = 1$. To see this, I first use the functional equation with x = 0 to get

$$\exp_b (0+0) = (\exp_b 0) (\exp_b 0),$$

and since 0+0 = 0,

$$\exp_b 0 = (\exp_b 0) (\exp_b 0) = (\exp_b 0)^2 .$$

I have narrowed $\exp_b 0$ down to either 0 or 1 since these are the only two numbers which satisfy the relation $a = a^2$. In fact, if a is <u>not</u> zero, I can divide by it and I immediately do so in $a = a^2$ to get a = 1. So a is either 0 or 1. Is it possible that $\exp_b 0 = 0$?

If $\exp_b 0 = 0$,

then $\exp_b x = \exp_b (x+0) = (\exp_b x) (\exp_b 0) = 0$

and $\exp_b x$ is equal to 0 for all values of x. But I have assumed that $\exp_b$ is not the zero function, which means $\exp_b x$ is non-zero for at least one value of x. If $\exp_b 0 = 0$, I am led to the absurdity that $\exp_b$ is the zero function, so $\exp_b 0$ must be unequal to zero. Since $\exp_b 0$ must be 0 or 1 and it can't be 0, I conclude that $\exp_b 0 = 1$.

The functional equation and the assumption that $\exp_b$ is not the zero function now leads to the fact that $\exp_b x$ is never zero. I will suppose that $\exp_b a = 0$ for some number 'a' . Letting x be any number and using the functional equation,

$\exp_b (x) = \exp_b (x - a + a) = (\exp_b x -a) (\exp_b a) = 0$,
and
$\exp_b (x) = 0$ for all values of x.

The assumption that $\exp_b a = 0$ for some number 'a' has led me to the absurdity that $\exp_b$ is the zero function. This means to me that $\exp_b a$ can't be zero for any value, a.

Further, since I can write any number, a, as a = a/2 + a/2, it must be that

$$\exp_b(a) = \exp_b(a/2 + a/2) = [\exp_b(a/2)]^2$$

and $\exp_b$ a is always positive since squares of non-zero numbers are always positive. Well, I like that. I like the idea of a function always being positive. The Pollyanna of the Ideal World, so to speak.

Noting that 0 = x + (-x), I can use the functional equation to show that $\exp_b(-x) = 1/\exp_b x$.

I can evaluate the rule for some numbers other than zero. For example

$$\exp_b(1 + 1) = \exp_b 2 = (\exp_b 1)(\exp_b 1) = (\exp_b 1)^2$$

and generally,

$$\exp_b n = \exp_b(1+1+1+...+1) = (\exp_b 1)(\exp_b 1) ... (\exp_b 1) = (\exp_b 1)^n$$

I see that exp 1 is going to be deeply involved in this and I'm glad I have a name for it; **exp 1 = b**. At this point in the argument b can be any positive number and for any value of b I choose, I get an exponential function.

$$\exp_b 2 = b^2$$

and

$$\exp_b n = b^n .$$

Using the fact that 1 = 1/m + 1/m + ... + 1/m it follows that

$$\exp_b 1 = \exp_b(1/m + 1/m + ... + 1/m) = \exp_b(1/m) \times \exp_b(1/m) \times ... \times \exp_b(1/m),$$

$$b = (\exp_b 1/m)^m$$

and

$$\exp_b 1/m = b^{1/m}.$$

A combination of these arguments shows that

$$\exp_b(n/m) = b^{n/m}.$$

I now have a rule for $\exp_b$ that works for every rational number, r,

$$\exp_b r = b^r .$$

Here I am again with a formula that is true for rational numbers and I am unabashedly going to assume that the formula,

$$\mathbf{b^x = \exp_b (x),}$$

works for every real number, x. I don't really like this step and I have some misgivings about it, but I like the thought that the formula doesn't work for irrational numbers even less.

It would seem that I have started with a functional equation and the condition that $\exp_b$ is not the zero function, and ended with an explicit formula for the rule of the function, $\exp_b$. This is a little deceiving and there are a few snags.

What is the meaning of b^x if x is an irrational number? This is the same problem I had when I defined the rule for the power function. I don't have a way to even approach finding that number; I have no prescription that gets me from b to b^x .

And if I did know the way to the number, b^x , would it equal $\exp_b x$ for x irrational as it did for x rational?

I am going to deal with the first snag by just assuming that there is a prescription to find b^x and that if I really wanted to, I could find it in some book. And, yes, whatever that prescription is for b^x , it gives $b^x = \exp_b (x)$.

The functional equation now becomes

$$b^{x+y} = \exp_b (x + y) = (\exp_b x) (\exp_b y) = b^x b^y$$

which is a standard law of exponents.

The relation, $\exp_b (-x) = 1/ \exp_b x$, becomes $b^{-x} = 1/ b^x$.

I have evaluated the rule for only one number, exp 0 = 1, but because I have shown that $\exp_b x = b^x$, I have given the rule, $\exp_b x$, all the laws and structure of exponents.

The function $\exp_b$ was defined by a functional equation and a condition. The number b^x was independently defined somewhere else, anxiously waiting no doubt, to be used as the rule of some function. It appears that $\exp_b$ is its true love.

In the function $\exp_b x = b^x$, the exponent is the variable and the base is constant, as opposed to the power function , $f(x) = x^a$, where the exponent is constant and the base is variable.

There is a value for b that makes using calculus with this function a true pleasure. The value has the name 'e' and it is an irrational number. Its first three digits are 2.71... .It was named 'e' after a man by the name of Euler and it first arose in a compound interest problem which he posed. If I put a dollar in the bank at 100% interest, compounded n times during the year, then at the end of a year I have $(1 + 1/n)^n$ dollars. Euler wondered what would happen if the interest was compounded continuously during the year, that is, what would you get if you let n go to infinity. The answer was the number 'e'. The number 'e' is defined as the number that $(1 + 1/n)^n$ gets close to as n gets larger and larger.

If I use this value for b, I drop the subscript and write 'exp x' instead of '$\exp_e x$' .

$$\mathbf{exp\ x = e}^x$$

The next two functions that I want to consider are trigonometric functions, **sine** and **cosine,** which are abbreviated **sin** and **cos**. The functions sine and cosine satisfy the functional relations

$$\sin(x + y) = (\sin x)(\cos y) + (\cos x)(\sin y)$$
$$\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y)$$
$$\sin(\pi / 2 - x) = \cos x.$$

These relations tell how sine and cosine deal with the addition of numbers in their domain. I get the feeling from these equations that sine and cosine are joined at the hip.

Since any two functions that were identically zero would satisfy the above equations, it is necessary to suppose that at least one of them, say cosine, is not identically zero. There is nothing here that would seem to restrict the numbers that these rules can deal with, so I would suppose that their natural domains are the set of all numbers.

The first thing I'm going to do is find sin 0. Using zeros for x and y in the first two equations, I get

$$\sin 0 = 2 (\sin 0)(\cos 0)$$
$$\cos 0 = (\cos 0)^2 - (\sin 0)^2 .$$

If sin 0 were not equal to zero, I would divide by it in the first equation, and see that

$$\cos 0 = 1/2 .$$

If I put this value in the second equation, I get

$$(\sin 0)^2 = -1/4,$$

which is absurd because no real number squared can be negative. Thus **sin 0 = 0**.

Using the functional equation for cos(x + y) with y = 0, it follows that

$$\cos x = (\cos x)(\cos 0).$$

If cos 0 = 0, it would follow that cos x = 0 for all values of x, which is absurd in the face of the fact that cosine is not identically zero. So, $\cos 0 \neq 0$

Since $\cos 0 = (\cos 0)^2$ and $\cos 0 \neq 0$, it follows that **cos 0 = 1**. This is the same type of argument I used when I determined that exp 0 = 1.

As an aside I might mention again that whenever I know a number is not zero, I divide by it. There is no other reason for knowing a number is not zero. I make this statement often and with no regrets.

I have, with the application of some ingenuity, now computed cos 0 = 1 and sin 0 =0.

Since $\sin(\pi / 2 - x) = \cos x$, it follows that

$$\cos(\pi / 2 - x) = \sin(\pi / 2 - (\pi / 2 - x)) = \sin x.$$

Consequently,

$$\cos \pi / 2 = \sin(0) = 0 \text{ and } \sin \pi / 2 = \cos(0) = 1.$$

I have two more notches in my gun,

$$\mathbf{\cos \pi / 2 = 0 \text{ and } \sin \pi / 2 = 1.}$$

Writing $\pi / 2 = (\pi / 2 - x + x)$ and using the functional equation for sin (x + y), I get what, in my opinion, is one of the most important relations in mathematics,

$$1 = \sin \pi / 2 = \sin x + (\pi / 2 - x) = (\sin x)(\cos \pi / 2 - x) + (\cos x)(\sin \pi / 2 - x)$$

$$= (\sin x)(\sin x) + (\cos x)(\cos x)$$

$$\mathbf{1 = (\sin x)^2 + (\cos x)^2 .}$$

Using arguments of about the same order of difficulty as those I have been using, I can show that sin(-x) = - sin x and cos(-x) = cos x. For example,

$$\sin(-x) = \cos(\pi / 2 + x) = (\cos \pi / 2)(\cos x) - (\sin \pi / 2)(\sin x) = -\sin x$$

So far I have evaluated the rules of sine and cosine only for the numbers 0 and $\pi / 2$. I can also evaluate them at $\pi / 2$, π, $3\pi / 2$, and 2π without a whole lot of trouble, but as I have said before, I am not so much interested in the specific numbers returned by the rules as I am in the general character of the rule. The functional equations go to the heart of the character of the rule. Calculus is not about numbers, it's about character, so from the point of view of calculus, defining sine and cosine from the functional equations is OK. **Calculus is about functions, not the numbers they produce.**

In the case of sine and cosine I have avoided all mention of angles, radians, and all that stuff, so what I have done can't be all bad. I say all this realizing that eventually someone, somewhere, sometime has to deal with the numbers and apply them.

The last basic function is the **natural logarithm** and is symbolized by **ln**. The functional equation satisfied by the natural logarithm is

$$\ln xy = \ln x + \ln y.$$

I am also going to impose three conditions:

1. 1 is the only number that the rule associates with 0
2. if a > 1, then ln a > 0
3. ln e = 1, where e is the number $(1 + 1/n)^n$ gets close to as n goes to infinity.

It is moderately interesting that the functional equation for ln is the same as that for exp with addition and multiplication interchanged. The arguments that I use to find out more about the rule of ln have a flavor similar to the arguments I used with exp.

If I let x = 1 and y = 1 in the functional equation,

$\ln 1 = \ln 1 \times 1 = \ln 1 + \ln 1$, and so **ln 1 = 0**.

When showing exp 0 = 1, I looked at $a = a^2$ which had 0 and 1 as its only solutions. When showing ln 1 = 0, I looked at a = a + a = 2a, which has 0 as its only solution.

If I write 1 = x (1/x), and use the fact that ln 1 = 0, I can show that

$$\ln (1/x) = -\ln x$$

and

$$\ln x/y = \ln x - \ln y.$$

This is like the idea of writing 0 = x + (-x) when working with exp.

The second condition anticipated that ln 1 = 0 and states that 1 is the only number that ln associates with 0. This is generally true. If ln x = a, then there is no other number, y, that ln associates with a. In other words, the only way ln x = ln y = a can happen is if x = y.

I would like to justify this last remark. If ln x = ln y, then ln x - ln y = 0. I can now use some of the things I have already shown about ln and get

$$\ln x - \ln y = \ln x/y = 0.$$

But 1 is the only number ln associates with 0, so x/y =1 and this only happens when x = y. I have shown that if ln x = ln y, then x = y. If x = y, then ln x = ln y because ln is single valued. A compact way of saying all this is that **ln x = ln y if and only if x = y.**

The function ln is increasing. That is , if a > b, then ln a > ln b. To see this, I first note that if a > b, then a = cb where c >1. I now use the functional equation to get

$$\ln a = \ln cb = \ln c + \ln b.$$

I know from the second condition that since c > 1, I have ln c > 0 and

$$\ln a = \ln cb = \ln c + \ln b > \ln b.$$

Putting all of this together,

a > b implies ln a > ln b.

I have to give some thought to the domain of this function. I don’t see any problems when the rule is applied to positive numbers. Of course, I don’t know the rule explicitly and problems may arise as I move closer to that knowledge, but for right now, positive numbers look fine.

The negative numbers are a problem. The computation

$$0 = \ln 1 = \ln(-1)(-1) = \ln(-1) + \ln(-1) = 2\ln(-1)$$

leads me to ln(-1) = 0, which is absurd because 1 is the only number that ln associates with 0. Since every negative number has a multiplicative -1 lurking in it, I think it best to leave the negative numbers out of the domain of ln.

The stark truth that $0 \times 0 = 0$ would seem to imply that ln 0 = 0. Alas, the same problem. It looks like 0 doesn’t make the cut either. The **domain of ln is the set of positive numbers.**

I have made little attempt at evaluating the rule for specific numbers in the domain but have spent my time seeing how the rule behaves generally. I continue in this vein.
In a way that is reminiscent of the way I dealt with exp, I write

$$\ln x^2 = \ln xx = \ln x + \ln x = 2 \ln x$$

and generally

$$\ln x^n = \ln (xx \ \ldots \ x) = \ln x + \ln x + \ldots + \ln x = n \ln x.$$

Following my nose,

$$\ln x = \ln x^{1/2} x^{1/2} = \ln x^{1/2} + \ln x^{1/2} = 2 \ln x^{1/2}$$

and

$$\ln x^{1/2} = 1/2 \ln x.$$

If I write $x = x^{1/n} \ x^{1/n} \ldots x^{1/n}$, I can generalize the above to

$$\ln x^{1/n} = 1/n \ \ln x.$$

I can combine the relation for integers with the relation for fractional exponents and find

$$\mathbf{\ln x^{n/m} = n/m \ln x.}$$

The relation $\ln x^a = a \ln x$ holds true for any rational number, a, and as before I make the leap and assume that

$\ln x^a = a \ln x$ for any real number a.

The first time I made the transition from a formula being true for rational numbers to the formula being true for all numbers, I was a bit nervous, maybe even afraid. Now I do it without a thought. The oftener a crime is repeated, the easier it gets.

I have assumed that ln satisfies a functional equation and three conditions:

$\ln xy = \ln x + \ln y$,
$\ln a > 0$ if $a > 1$,
$\ln a = 0$ if and only if $a = 1$.
$\ln e = 1$

Using only the <u>first three facts</u>, I have come up with a bunch of other relations that ln also satisfies.

$\ln 1 = 0$
$\ln x = \ln y$ if and only if $x = y$
$a > b$ implies $\ln a > \ln b$
$\ln (1/x) = - \ln x$ and $\ln x/y = \ln x - \ln y$.
$\ln x^a = a \ln x$ for any real number a.

It turns out that there are an infinite number of functions whose rules satisfy the functional equation and the first two conditions. Of course all of these functions also satisfy the five derived relations. It is now that I use the condition, ln e = 1. I can see that a person might wonder why I made the choice, ln e = 1. After all, e is not the most straight forward number that ever came down the pike. I don't have any idea, at the moment, of what e is numerically. As a matter of fact, I can see why a person might wonder where any of the conditions came from.

Well, I took the functional equation from the fact that ln xy = ln x + ln y is the reason why logarithms were invented in the first place, to change a multiplication problem into an addition problem. Multiplying two six digit numbers by hand was a formidable task before the advent of the computer and tables of logarithms made it possible. This use of the logarithm has passed into history and the importance of ln now lies more in its close relation to the exponential function. But regardless of the diminished importance of the functional equation in today's world, that equation is still the heart of ln.

When writing this opinion, I started with the functional equation and set out to derive five of the laws that ln satisfies. I needed two facts to do this and I went back and made them given conditions so that I had them when I needed them.

The choice of ln e = 1 has to do with the close relation of ln to exp and I, personally, think that exp is the archangel of functions.

Now, exp x = e^x , and ln has a nifty law that deals with exponents. If I apply this law,

$$\ln \exp x = \ln e^x = x \ln e$$

and since ln e = 1,

$$\mathbf{\ln \exp x = x.}$$

<u>This is why I chose ln e = 1.</u>

This is an interesting turn of events. It appears that ln undoes what exp does to x.

Does exp undo ln? Well, I'll let y = exp(ln x), that is, y = $e^{\ln x}$. If I apply ln's rule to y, I get

$$\ln y = \ln e^{\ln x} = (\ln x)(\ln e) = \ln x$$

and

$$\ln y = \ln x.$$

But this can only happen if x = y, or

$$y = \exp(\ln x) = e^{\ln x} = x$$

and it appears that exp does undo ln. I have two relations that completely bare the intimate relation between exp and ln,

$$\mathbf{\exp(\ln x) = e^{\ln x} = x}$$
$$\mathbf{\ln \exp x = \ln e^{x} = x}$$

Without even thinking about it, I assumed that the range values of ln were in the domain of exp when I considered exp(ln x), and the other way around when I considered ln (exp x). In the heat of battle I tend to suppose that things are the way I want them to be just because I want them to be that way, an arrogant kind of attitude at best. In this case I am justified. While it is not particularly obvious, the range of ln is exactly the domain of exp, namely, the set of all numbers The range of exp is exactly the domain of ln, namely, the set of positive numbers. The pair of functions ln and exp form a marriage made in heaven.

When something this special happens, it must be given a name. **exp and ln are called inverses of each other.** The idea is completely analogous to 2 and 1/2 being multiplicative inverses of each other.

There is only <u>one</u> function that satisfies the functional equation <u>and</u> the <u>three conditions,</u> and that is ln.

I suppose that f is a function whose domain is the set of positive numbers and whose rule satisfies

1. f(xy) = f(x) + f(y)
2. f(x) > 0 if x > 1.
3. f(a) = 0 if and only if a = 1
4. f(e) = 1.

Then $f(x^2) = 2f(x)$, and generally, $f(x^n) = n\, f(x)$. Continuing in the same way I did for ln, I get $f(x^a) = a\, f(x)$. In particular, $f(e^{\ln x}) = (\ln x)\, f(e) = \ln x$. But I have just seen that $e^{\ln x} = x$, so f(x) = ln x. The function f was really ln. Any continuous function that satisfies 1., 2., 3., and 4. must be ln. I slipped in the word 'continuous' because that is the property of Ideal World functions that allows the transition from rational numbers to irrational numbers for the validity of the rule.

There are actually two other popular choices for the number ln associates with 1, ln 10 = 1 and ln 2 = 1 , but then the function is not called ln anymore; it is called log in the first case and $\log_2$ in the other. There is actually an infinite family of logarithmic functions.

At last, I consider the ways to combine these five functions. The first four methods of combination arise from the four arithmetic operations, addition, subtraction, multiplication, and division.

I start with two functions, f and g. A function is a rule and its domain, so I must give the rule and domain of the new functions.

The first new function is f + g. The rule of f + g is defined as (f + g) (x) = f(x) + g(x) and the domain of f + g is the set of numbers that the domains of f and g have in common.

The second new function is f - g. Its rule is defined as f - g (x) = f(x) - g(x) and its domain is the set of numbers that the domains of f and g have in common.

The third new function is fg. Its rule is defined as fg(x) = f(x) $\times$ g(x) and its domain is the set of numbers that the domains of f and g have in common.

The forth function is f/g. Division is a little different because the possibility of division by zero must be avoided. The rule of f /g is f /g (x) = f(x) / g(x) but the domain of f /g can't have any numbers that make g(x) = 0. These numbers are removed from the set of numbers common to the domains of f and g to get the domain of f/g.

There is a fifth way to combine two functions. If the domain of a function f contains part of the range of a function g, then the rule of f can be applied to numbers in the range of g that are also in the domain of f. In particular, the rule of f can be applied to numbers of the form, g(x), when they happen to be in the domain of f. I define a new function $f \circ g$ called the **composition of f and g**. Its rule is given by $f \circ g$ (x) = f(g(x)) and its domain consists of those numbers in the domain of g such that g(x) is in the domain of f.

Any kind of rule that I write down,

$$(\sin x + \cos e^{x}) / \ln x + x^{2} + 1$$

for example, is just a succession of applications of the five ways to combine functions to the five basic functions.

Functions form a set of objects in the Ideal World. What the Ideal World does with a set of objects is to try to put some kind of algebra on it, that is, devise some way to add, subtract, multiply and divide these objects. Generally this is only partially successful, but in the case of functions in is almost completely successful.