CHAPTER 2

LECTURE 2-1

I was sitting in the roof garden of my penthouse, looking out across the city from my sixty stories of space, when a movement on the rooftop of the building directly across from me caught my eye. A slim figure poised on the edge, bent its knees and jumped into a perfect swan dive...

Now is the time to construct the model of the falling rock in the Ideal World.

There are two aspects of the model; one is geometric and the other is dynamic. The geometric part is the path of the rock in space and the dynamic part is how the rock moves along the path. The path of the rock is there before, during and after the event, whose life is all too brief. The rock could have been shot out of a cannon or lowered on a string along the same path; the path would be the same but the dynamics would be different.

On the basis of my extensive observations of the descent of the rock, I make conjectures about the Real World properties of this motion. These conjectures may not be quite what I really, down deep inside, believe, because they have to be tempered with common sense.

Trying to model exactly what I feel to be true may lead to extreme complication and impossible computations. There also may be several quite different ideas of truth about the same phenomena, each one appropriate in the right context. If I put a lens in front of light it appears to be a wave, if I put a photo electric cell in front of light, it appears to be a particle. The way I model light depends on what I'm doing with it.

Regardless of how these conjectures come into being, I make them part of the axiom system of the model. In the model, they are no longer conjectures, they are truth and the only motion I'm considering is motion that satisfies the axioms. In the Ideal World of the model I don't have to justify that the motion has these properties, because I made the motion have these properties. I study the motion in the Ideal World model where the conjectures are assumed true and see how it compares with the corresponding Real World motion by experiment.

After I have entered my conjectures into the model as axioms, I will go to the laboratory and get Real World data on how long it takes a rock to fall various heights. On the basis of this Real World data and the axioms I will try to find a relation between the heights and the times which, with a little luck, can be expressed as a Real World formula.

The Real World formula is put into the Ideal World model to describe the dynamics and at this point, I should be able to predict how long it takes the rock to fall a given height. But, as my grandmother used to say, "There's many a slip twixt the cup and the lip."

What are these conjectures that I make about the Real World that are going to become axioms in the Ideal World? There is so much interplay between the path and the motion that it is hard to know where to start. The rock has to be dropped for the path to be known, but after the path is known, I don't need the rock anymore. I can go back and look at the path anytime I want, but if I want to look at the motion again, I have to drop the rock again.

On the other hand, it is the motion that I want to study and the path is of interest only because it arose from the motion of the rock. If there is a conflict between modeling the motion and the path, the motion rules. I will look at motion first.

My immediate aim is to find out how long it takes a rock to fall a given distance without having to drop the rock. This is plausible because, after some experimentation, I have come to the conclusion that given a Real World height there is a unique Real World time that it takes the rock to fall from that height. For that matter, it also seems to be true that if a rock is dropped and falls for a given amount of Real World time, it always falls the same Real World distance. While this remark is not really relevant to my immediate problem, it seems interesting and I will remember it.

The property that one set of physical quantities, in my case the single physical quantity, height, uniquely determines another set of physical quantities, in my case the single physical quantity, time, is a property of 'dropped rocks' that is shared by a lot of other Real World physical processes. It is this kind of process that I am exemplifying by dropping a rock. Whether a process has this property or not is determined by experiment.

Since I am representing physical quantities with Real World numbers, I think that the number that represents height must uniquely determine the number that represents time. Generally, the set of numbers that represent one set of physical conditions, which I think of as the initial state or initial condition, uniquely determines another set of numbers that represent another set of physical conditions at a different time or place.

These ideas fall under the **'principle of uniqueness'**. The use of this principle requires a little care.

Suppose I drop the rock sixteen feet from the ground. There is a unique time when the rock is twelve feet from the hand that drops it, or at any distance less than sixteen feet. There is no time when the rock is at a distance greater than sixteen feet from the hand. It is at a distance of sixteen for a long of time. Until somebody picks it up, actually. Uniqueness is gone at sixteen feet and with a vengeance. The 'principle of uniqueness' applies for distances less than sixteen feet but not for sixteen feet.

If I look at it the other way around, given a time there is a unique distance between the rock and the hand. For times longer than one second, the distance is sixteen feet. The 'principle of uniqueness' applies for all values of time.

The 'principle of uniqueness' claims the uniqueness only one way. In my particular example, not only does a height between zero and sixteen determines a unique time but a time determines a unique distance. That is nice but not really needed. My problem is "given a height, find a time", so the uniqueness that is essential for me is the unique time determined by the height. The distance between the rock and the hand does not determine a unique time when the distance is sixteen feet, so if I look at the problem in these terms, I must restrict the distances to be less than sixteen feet.

In problems like this I have or am given one quantity which I think uniquely determines another quantity. Since the choice of the value of the given quantity does not depend on anything except the ‘principle of uniqueness’, which must be satisfied, the given quantity is said to be **independent**. The quantity that results is called **dependent**.

I think that the idea that one thing uniquely determines another is so important that I have a ‘felt need’ to say it over and over. It is my opinion that it is the nature of this planet, in particular, the abundance of physical laws where one thing seems to uniquely determine another, that shaped the fierce determinism of mathematics. If I were asked to pick one phrase that I think typifies mathematics, I would certainly consider, "If this, then that."

There could be a planet that was entirely covered with water. On this planet nothing falls like a rock. Living under water takes away a lot of determinism. This planet follows a complicated orbit through a system of several suns. As a result, the length of days and nights are random within the fairly generous limits that days are not so long that everything melts and nights that are not so long that everything freezes. Life is difficult but possible. The magnetic fields from the suns make a shambles of any electromagnetic theory since the result of any experiment seems random. On this planet an entirely different idea of the relationship between physical quantities would arise which, I think, would give rise to a very different mathematics.

The dolphin lives underwater and its interest in electromagnetic phenomena is probably more practical than theoretical. It is no doubt aware that there is some regularity in night and day, not unlike the average person perhaps, but I don't see that it would make the impact in its life that it makes in mine. Tides might be of more interest. The mind of the dolphin evolved in the presence of a random physical world, an ample food supply, no natural predators and with the run of 2/3 of the planet. I wonder what kind of mind that might be.

As a terrestrial earthling, my model of the motion of an object will include the idea that given a distance there is a unique time that it takes to fall that distance. In this discourse I will only consider this type of physical process, where a given initial state has unique, well determined consequences. Since this includes almost all of mechanics and electromagnetics, it is not a serious restriction.

LECTURE 2-2

I walked to the edge of the garden to get a better view. The swan dive evolved into a double, pike position with a full twist...

Besides the intellectual observation about the very demanding relation between the height and time, there is a more aesthetic observation about the nature of the motion itself. Karate, ice skating and gymnastics are three examples where motion is judged aesthetically. How does the rock in motion score? This is the topic of the second assumption about the nature of the motion of the rock in the Real World.

When I watch a rock fall, the first word that comes to my mind is ‘smooth’. This is no herky-jerky motion. It ‘smoothly’ covers more distance each second it falls as it ‘smoothly’, rushes down to go ‘thunk’ into the dirt. There is nothing about the rock that is ‘one way’ one instant and ‘another way’ the next instant. A quality that the sports of football, baseball, and basketball have in common is the beauty of their respective balls in flight; ‘smooth’, elegant motion. A pleasant summer afternoon can be spent dropping a pocket full of rocks from a railroad trestle, watching them ‘smoothly’ fall into the river below.

This is the point of view of my eyes.

This smoothness of motion is a quality called **continuity**. Continuity is actually a point property, where the process changes smoothly from one state to the next. Time moves smoothly from one instant to the next and the rock moves smoothly from one point to the next. The entire process is continuous because it is continuous at each point. All of the processes of mechanics and electromagnetics are continuous in the sense that they change smoothly from one Real World state to the next.

A point of space or time where a process is not continuous is a **discontinuity** of the process, and as soon as the process has even one, it isn't continuous anymore. If, after 1 second, the rock is 16 feet from the hand that dropped it and the very next instant it is 20 feet from the hand, then the motion of the rock has a discontinuity when the time is 1 second after dropping. The process of dropping a rock would then be **discontinuous** instead of **continuous**.

The discontinuity in this case is a **jump** discontinuity. Continuous processes can't have any jumps.

Since the processes that are about my size are all continuous, it is hard to find an everyday example of something that jumps. An electron does not go smoothly into that next energy level, it jumps, but I seldom experience this directly in my everyday life.

I see the path of the rock as a vertical string of beads, each bead being a Real World point, and I see the interval of time as a string of beads, each bead being a Real World instant of time. As each instant of time occurs, its bead lights up; as the rock gets to a point, its bead lights up. Time always moves smoothly to the next instant with a steady beat. The time beads light up successively and none are left out, dark and unused.

The rock always falls smoothly to the next point of the path but with a quickening beat. The point beads light up successively and none are skipped and left dark and unvisited.

This is the second basic principle of motion, the **‘principle of continuity’**. It says that if the rock is at a point of the path, then it moves to the next point and doesn't skip any.

I assume that the ‘principle of continuity’ is true because when I watch an object move along a path, it looks like it is hitting all the points. My extended senses see this all the way down to the atomic level, but the decision to assume this principle was first made when measuring by eye was as good a way as any. I believe in my heart that the ‘principle of continuity’ is true because of the evidence of my eyes.

This is an example of believing that something is ‘Real World’ true and basing that belief on bad evidence.

A television camera samples motion every 1/30th of a second, and in doing an ordinary gesture a hand can move several inches in that time. If I run my finger along a line across the top of my desk, from one point to another, it looks as though my finger passes through every Real World point on the path. The first frame of a video tape catches my finger at the first point of the path. In the second frame my finger is instantaneously about 1 inch from where it was before, and in the third frame it is instantaneously an inch further yet.

On the video tape, my finger does not go to the next point when time goes to the next instant, it moves an inch, skipping over a lot of points. But, when I watch the tape, my finger appears to move smoothly across the desk, perfectly continuous. The smallest unit of time in the world I see on television is 1/30 second and except for that funny business with the wheels on covered wagons, I can't see a lot of physical difference between the world I see on television and the world I see out the window.

My eyes see motion that has a discontinuity every 1/30 of a second as continuous, hardly the basis for a comprehensive theory of motion. Still, I might go with a ‘continuous’ model even if the Real World wasn't any more continuous than a movie. My eyes give me a strong, grass roots feeling that the Real World is continuous and there is nothing wrong with using a continuous model for a discontinuous reality. Economics regularly uses continuous models in situations that are inherently discontinuous.

Eyes are easily fooled. That is why magicians can make a living and why the movies are a billion dollar business.

There are several ways to look at the ‘principle of continuity’ and there is one that is particularly useful. If the rock is at one point of the path at one instant of time and at another, lower point at another instant of time, then the rock has been at every point between them. The rock doesn’t skip any points, so for every point between the top and bottom points, there is a unique instant of time when the rock is at that point.

If there is a pencil across the intended path of my finger on the desk, I must either move the pencil or stop moving my finger. My pencil and finger can't be in the same place at the same time and since my finger occupies every point on its path to the other side of the pencil, there is a problem with the point that the path and pencil have in common. This is the idea behind the proof that mended fences keep cows in pastures.

The ‘principle of continuity’ is not special for objects in motion but holds for all the processes of mechanics and electromagnetics.

As a process follows its fated course, it doesn't skip any states along the way. If the process is in one state at one time, or place, and at another state at another time or place, then it will assume every state between the two.

I have been using two ideas more or less interchangeably. When the rock is at a point on the path, I can think of the time being associated with the point or I can think of the time being associated with the distance the rock has fallen to that point. I can think of time as the instant of time when the rock was at the point or as the length of time it took to get there. I will use which ever interpretation seems appropriate for my context.

LECTURE 2-3

I recalled that throughout all of the figures graceful motion, its center of gravity was smoothly following a parabolic arc; subject to the irregularities introduced by the fact that it was falling through air, of course...

I have my two basic assumptions about motion, uniqueness and continuity, in hand and now I can turn my attention to the path, which seems to be a segment of a straight line. I could model the path using a finite number of equal Ideal World line segments to model the Real World points on the path. This would make it a model of the Real World path I see with my measuring instruments. Or I could model the path as a segment of an Ideal World line with an infinite continuum of points, making it a model of the Real World path I see with my eyes.

Time has to be modeled too, but my model of time will follow the lead of what I decide to do for the path.

The fact that 'principle of uniqueness' must hold in the model gives me a little insight into the nature of the path with respect to the interval of time.

In the model, each distance is associated with a unique length of time, or each point on the path is associated with a unique instant of time in the time interval.

There is just one 'instant of time' associated with each point and if all the 'instants of time' are used, there can't be more 'instants of time' than points. In fact, if there are more 'instants of time' in the interval than there are points on the path, then either the rock is at some points for more than one 'instant of time', contradicting the 'principle of uniqueness', or there are 'instants of time' that are not associated with any point on the path and the rock would not be anywhere at those instants. The rock has to be somewhere at every 'instant of time' in the interval, so this is absurd. There can't be more 'instants of time' in the time interval than there are points on the path.

If there are fewer 'instants of time' in the interval than there are points on the path, then at least two points would have to be associated with the same 'instant of time'. Since the rock can't be in two different places at the same time, this is absurd and there can't be more points than instants. The fact that there can't be more points than 'instants of time' follows from the basic physical reality that things or people can't be two places at once, an idea that my mother tried to explain to me and I did not fully understand until I had children of my own..

I seem forced to conclude that there must be the same number of points on the path as there are 'instants of time' in the interval. The same argument implies that there are the same number of points in any part of the path as there are instants in the time interval it took the rock to fall through that part.

Since my best perception of the Real World is through my instruments, I'll try the discrete model first and use the analogy of the beads to see if continuity and uniqueness can maintain.

I don't see any problem with continuity. Continuity just says that no beads can be skipped. This seems a little strange because the object seems to be jumping from point to point but I have to remember that there isn't anything between the two points so I'm not jumping over anything. I think that I can model the path in space and the interval of time discretely and still keep the idea of continuity.

Can uniqueness exist in the discrete model? I run into a little problem here. The idea is that when I make the 'principle of uniqueness' an axiom, I am considering only the motions that satisfy the axiom. If I use 'instants of time' which all have the same positive length, and points which all have the same positive length, then there is only one motion where the number of 'instants of time' in the time interval is the same as the number of points on the path.

Suppose that time beads represent 0.1 seconds and point beads represent 0.01 foot. If the rock falls one foot, then there are 100 point beads on the path. Since there are the same number of points as instants, there must be 100 instants in the time interval which makes the time interval be 10 seconds. Each tenth of a second that ticks on the clock, the rock falls a tenth of a foot. This is the only motion that this model allows.

This is not how I remember the rock falling.

The problem is that **if a point bead has a positive length**, I can always move the rock so slowly that it will be at a point bead longer than one instant of time; and **if a time bead has positive length**, I can move the rock so quickly that it will pass two point beads in an instant of time. Neither of these possibilities is viable.

It may be possible to fix up the finite model, but before I try I think I'll see how a segment of a line from Euclidean geometry works. The points on a segment of a Euclidean line form a continuum of points, each of which has zero length.

In this case I will also model time as an infinite continuum of instants, each of which has zero duration. I have been thinking of 'instants of time' as beads on a string, so as I think of there being more and more, smaller and smaller beads going through a metamorphosis into an infinite continuum, it is rather natural to think of the infinite continuum of 'instants of time' to be a lot like points on a line segment.

If the points of space and the instants of time form a continuum, the model is called **continuous**.

If there are a finite number of points of space and instants of time, the model is called **discrete**.

Since I think of the Real World as being finite and discrete, it seems to be a little contrary that I must opt for a model that is neither finite nor discrete. I'm not going to model the Real World the way I think it is, I'm going to model it the way I see it.

It is not that a discrete model can't be made to work and work very well but it requires a lot of computational power. The decision to use the continuous model was made a long time ago when computation to five and six places was all done by hand. The idea of modeling a problem a point at a time by hand did not seem very appealing before the

twentieth century. It was also a time when the eye was one of the more accurate measuring devices and motion looked continuous then as it does today.
For whatever reason, the continuous model was done first. The discrete model is often built as an approximation to the continuous model and not developed from scratch.

Actually, I think the Real World is discontinuous, but that the discontinuities are very, very small. So small, in fact, that they can't be measured. I am going to confine myself to processes that have no measurable discontinuities, like the falling rock.

If a Real World process is discontinuous at a level where the discontinuities can't be detected, is it really discontinuous? I have noticed that people can be divided into two groups by a class of dichotomies. Keeping food separate on the plate or mixing it up is one such dichotomy. Moving immediately out of a freeway lane that is about to be closed or waiting until the last minute is another. Some people eat the best part first, and others save the best until last. How people answer the question, “Do unmeasurable quantities exist?”, may form another such dichotomy.

LECTURE 2-4

The figure flattened out like a sky diver, as if to catch its breath before going into a triple, tuck position. Velocity, acceleration, angular momentum; they were all there...

One consequence of choosing the continuous model is that the length of an instant of time and the length of a point are no longer positive. I am going to have to deal with the problem of line segments of length one being composed of points of length zero. The 2500 year old problem of how things of length zero can add up to something of length one appears.

In my opinion the problem is a straw problem. In the Real World, it is just not true; things that are zero, add up to zero. There is no such thing as a free lunch. I have the second law of thermodynamics which tells me that I don't get something for nothing.

But in the Ideal World, this seeming anomaly is true and I want to understand the intuition behind why it's true in the Ideal World. Since it is true in the Ideal World, there must be something in the structure of the Ideal World that supplies a reason for this truth. The thing that makes it hard is that there is no analogy in the Real World. In fact the idea is Loony Toons, literally, in the Real World.

When Alice went 'through the looking glass' she encountered truth that was different from truth on the Real World side. She had to walk in the direction opposite from where she wanted to go; she could eat a mushroom and change her size. In order to function in Wonderland, she had to find out what truth was and integrate it into the way she thought.

Carlos Castenada's description of learning about the world of sorcery and how to function within its truth, so unlike that of the Real World, reminds me of a student of mathematics learning about the Ideal World and how to function with its truth, so unlike that of the Real World.

Still, the Ideal World idealizes the Real World, so they have a lot of things in common too.

Why might a sum of zeros in the Ideal World add up to one? "Upon what meat does this our Caesar eat, that he is grown so great?"

The Ideal World's edge is that it has infinity. It not only has infinity, it has hierarchies of infinities. In the Real World I can only add up a finite number of zeros but in the Ideal World I can add up an infinite number of zeros, and if that is still zero, I can add up an even bigger infinity of zeros. Adding up an infinite number of anything can only be done in the Ideal World.

And what about the 'zeros' themselves? In the Real World there is one kind of zero but in the Ideal World there is zero and then again there is zero. There is something different about the zero that represents the number of a male calico cats and the zero that represents the length of an instant of time or the length of a point. In the Ideal World, the sum of the lengths of the points on a line one unit long is one. That is just the way it is.

I can try to understand this Ideal World phenomenon as I continue my investigation, but for right now I am going to have to live with it. I think that awareness of a phenomenon and its application comes first, and over a period of time, understanding follows. I think that people knew that copulation resulted in children and applied this knowledge long before they had much understanding of the biological processes involved.

My grandfather described a cribbage hand with no points as 'what the little boy shot at', where I would describe it as 'zero'. He was looking at the hand set theoretically; the set of points was empty, while I was using the cardinal number that described the number of elements in the set of points in the hand.

LECTURE 2-5

The figure was just a dot, far below me. I walked back to my chair in the garden and sat down to watch the sun set behind the city. The motion of the bodies in the solar system has always interested me...

I now want to see how the ‘principle of uniqueness’ and the ‘principle of‘ continuity’ work in the continuous model.

The ‘principle of continuity’ says the rock doesn't miss any points and I don't at the moment see any problem with this assumption.

Is the problem with uniqueness resolved? The problem in the discrete case was that there was not the same number of instants of time as there were points on the path. Now the path is an infinite continuum of points and the time interval is an infinite continuum of instants. Again, the hand of infinity blesses my efforts. Two such infinite continuums have the same number of points.

One of the ideas about infinity that is a little hard for a Real World person to swallow is that there are the same number of points in an Ideal World segment that is one inch long as there are in one that is two inches long. While the mechanism is not entirely clear, I can at least try to justify my remark in the special case.

First, I have to decide what it means for two sets to have the same number of elements. I know that I have the same number of digits on each hand because I can put my hands together matching the fingers and thumb of my left hand with the fingers and thumb of the right hand. I can make a one to one correspondence between the digits on each hand.

I will consider two sets to have the same number of elements if there is a one to one correspondence between the elements of the sets.

If I take a number in the one inch interval and multiply it by two I get a number in the two inch interval. This process associates every number in the one inch interval with a unique number in the two inch interval.

The process associates a number in the one inch interval with <u>every</u> number in the two inch interval. Indeed, if I have any number, n, in the two inch interval and divide it by two, I get a number, m = n/2, in the one inch interval which the process associates with the original number, n, in the two inch interval. Every number in the two inch interval is the result of applying the process to a number in the one inch interval.

What the process does is tie a string from every point in the one inch interval to exactly one point in the two inch interval and every number in the two inch interval has a string tied to it. I have a one to one correspondence between the numbers in the one inch interval and the two inch interval. They have the same number of elements.

This seems to contradict the well known fact that the whole is greater than the part. Well, in the Ideal World this is not always the case. I don't have to understand why it is true in the Real World because it isn't true there. I want to eventually understand why it is true in the Ideal World.

But regardless of why it is true, it is true, and the problem that I had with the discrete model is no longer with me. Other problems with this model may arise, but for now I am ready to proceed. “Damn the torpedoes and full speed ahead.”

LECTURE 2-6

The person who made this marvelous dive must also have been interested in motion and I felt a deep kinship. I hoped for a momentary suspension of the laws of nature...

I feel that I must be careful to distinguish between the Real World and the Ideal World. I know a person who believes firmly in monarchy. He explains to me in detail why it would work better than other forms of government. If people were the way he thinks they are, he could well be right, but I am afraid that his people live in the Ideal World where it is much easier to get governments to work.

I grew up believing that $\sqrt{2}$ was a number in the Real World. I thought that it was the length of the diagonal of a square of side one, forgetting that I had never seen a square of side one. In fact, I had never seen a segment whose length was exactly one and I had never seen an angle that was exactly right. Nor have I to this day.

In the Ideal World the number 1.41421 is close to $\sqrt{2}$ and since 1.41421 is a Real World number, its proximity to $\sqrt{2}$ seemed to drag $\sqrt{2}$ into the Real World. I forgot that 'close' isn't the same as 'equal' and that 'knowing about what something is' is not the same as 'knowing exactly what something is'.

The farther I go in my analysis of motion, the more this analysis will be in the Ideal World. I will talk more in terms of the Ideal World model and less in terms of the actual Real World problem. It is easier in the Ideal World. If it wasn't easier in the Ideal World, it would seem perverse to model it there.

If I want to make a truss in the shape of an isosceles right triangle where the equal sides of the triangle are ten feet, then I will make the third side 14.14 feet long. This is a Real World problem as solved by a Real World carpenter. But in the Ideal World, I think of the third side as being $10\sqrt{2}$. In the Ideal World I can ask The Scarecrow to cut me a board $10\sqrt{2}$ feet long.