CHAPTER 1

LECTURE 1-1

If I add the age of the jockey to the age of the horse...

For thousands of years it has been extremely important to some human beings to understand the physical world around them. The motivation behind their desire ranged from idle curiosity to an urge to use this understanding to control the physical world. I do not know what went on in the minds of *A. afarensis* but I can believe that there were those among them who wondered at the world around them and maybe even thought about it a little. Curiosity is old.

An important aspect of this understanding is prediction. In particular, these people want to be able to predict what will happen next if they know what is happening now; or what is happening there if they know what is happening here. I have always found it interesting that Kepler was an astrologer and was hired to predict future events, not come up with laws of the universe, which predict events also but probably not the ones his boss had in mind.

The process of trying to attain this knowledge gave rise to a language particularly well suited for the description of the physical world. This language was remarkable in that not only could it be used to describe the physical world, it could be manipulated to give conjectures about this world that were generally true. I must be a little careful about using the word ‘true’. Truth in the physical world is slippery concept at best. I can say that the prediction of physical events using this language has been successful.

My purpose is to consider how a certain part of that language, calculus, might naturally arise out of a quest for some understanding of the physical world around me. I want to describe the behavior of objects in motion, like planets and baseballs and falling rocks, and in so doing, I hope to develop a language that both describes the motion of the object and can be used to predict its future behavior. The language should be able to describe the path of a cannon ball and predict when and where it will land.

I’m going to focus on a rock and the motion that results from dropping it. I want to know what the rock is going to do after I drop it.

I choose to look at a falling rock for several reasons. I am familiar with falling objects and observe them daily. Rocks typify a certain group of objects, like toasters and catsup bottles, all of which seem to fall in pretty much the same way. I also think that the problem of the falling rock has some general significance. The motion of objects in space has been of interest to the human animal for a long time, a specific point of interest being to hit something at a distance with the object. The intellectual distance between dropping a rock and hitting the ground and launching a rocket and hitting an orbit around the moon, is surprisingly small.

I find a rock that fits nicely in my hand, hold it out in front of me and drop it. Based on my knowledge of the rock at the instant it was dropped, I want to have knowledge of the rock after that instant for as much time as possible.

The expression, ‘knowledge of the rock’, is a bit vague. After all, the rock has a chemical composition and a crystal structure and these properties might be considered part of the ‘knowledge of the rock’. My interest centers on the motion of the rock and I leave other aspects of the rock to other disciplines.

I observe what happens after I drop the rock using my eyes, ears, time pieces, tape measures and whatever else seems useful. On the basis of these observations I develop a way to describe what I observed.

I have to decide where to drop the rock. For example, I could drop it down a deep hole, I could drop it to the ground I am standing on; I could drop it from the roof of a two story apartment building; or I could drop it from someplace quite high like one the Trade Towers in New York City. Seeing no compelling reason to chose one of these experiments over the other, I will do all four.

‘Knowledge of the rock’ will be expressed in terms of symbols and the first symbols I’m going to use are words. Here is my first try.

In each case the rock is motionless at the instant it is released. This is the initial condition of the rock. Then the rock falls. In the long term, the rock hits the ground and is then motionless. In the very long term, the sun goes red giant and vaporizes the rock. The future of the rock seems accounted for and the description is independent of the spot where I dropped it.

For large parts of humanity this description is good enough. Before the invention of clocks, the day was the basic measure of time and this general kind of description was the only description there was. It is not meaningful to tell someone how long it took the rock to fall in days.

As I think about it, though, I did observe that the rock took longer to hit the ground if it was dropped from a higher spot. Even if I can’t measure how much time it took to fall, I can tell that it takes longer to fall from one spot than another. This is information that seems interesting enough to put in the description. I will refine my first description.

The rock is held motionless in my hand. The instant that the rock is released it begins falling toward the earth. It falls until, after a very short amount of time, it hits the earth and is again motionless. The higher the point of release, the longer it takes to fall to earth. I think I'll leave out the bad news about the sun going red giant.

For someone measuring time in days, this is a reasonable description. All the times are short in terms of days and it captures the idea that it takes a longer time to fall a longer distance.

There was a slight problem with one of these experiments. In two of the cases I could see the rock hit the ground. When I dropped the rock down the hole, I could hear it hit bottom. But I could neither see nor hear the rock hit the street when I dropped it off one of the Trade Towers. How do I know that the rock ever got to the street? Perhaps it stopped somewhere above the seventh floor.

The fact is, I have dropped lots of rocks in my life and I feel sure that it made it to the street. Unfortunately I could not think of a way to sense the moment it hit the street so I will drop that part of the experiment.

This brings up the question of how much I can rely on past experience to predict the outcome of an experiment. I, personally, am inductive. If red comes up twenty times in a row on a roulette wheel, I would bet that it comes up red on the twenty-first. On the other hand, I know people who would not feel sure that the rock dropped from the top of one of the Trade Towers did hit the ground, no matter how many rocks they had dropped before. This might be the time when it didn't fall all the way to earth. They would tend to bet on black.

LECTURE 1-2

T he problem was, I didn't know exactly when either the horse or the jockey was born...

Now, it is not that people who measure time in days do not have deep and subtle things to say about falling rocks, but my interest requires a finer subdivision of time. As a boy of nine or ten I lived next to the Payette River in the mountains of Idaho. There were some high granite boulders above the river and on the Forth of July the drill was to throw fire crackers from the boulders so that they exploded just as they hit the water causing little geysers. To this end, I timed how long it took a fuse to burn and how long it took the fire cracker to fall to the river. I could then light the fuse and hold the firecracker until the time left on the fuse equaled the time to fall to the river.

This method of throwing firecrackers required a unit of time that could conveniently measure the time for a fuse to burn and for a fire cracker to fall into the river. The unit of time that worked was the second: the fuse took three seconds burn, the fire cracker took two seconds to fall. These were numbers I could deal with. Doing this computation in days would have been awkward at best. Of course ‘awkward’ does not mean impossible. I might add that my method did not work all that well, and I attribute this to a lack of quality control in fuse manufacture. I did learn a lot about the care of burns.

In any event, time is certainly an essential part of my description of a falling rock, as it is an essential part of the statement of the problem: How does knowledge of the rock at one time determine knowledge of the rock at another time? My previous descriptions were in large part shaped by the assumption that I measured time in days. The final description will be shaped by my actual measurement of time, so I guess I think a little bit about that measurement.

What is time? I don't **know** what time is, but I do have an opinion.

Time is what I learned how to ‘tell’ as a child. Time is what everybody ‘knows’ it is. Time passes and I know what is meant by an interval of time. Whether an interval of time is long or short depends on who or what I am. I don't suppose that a second means much to a rock and a nanosecond, that is, a billionth of a second, doesn't mean much to me. I don’t think 10^{-40} seconds means much to a computer but a time of 10^{-40} seconds might mean a lot to a very young universe in the midst of a big bang.

Actually, I don't know how important 10^{-40} seconds is to the universe, young or old. An electron seems pretty small compared to the universe, so small that I might think that the universe would consider a single electron unimportant. I don't see how losing just one electron could harm the universe much. The universe, however, evidently feels that each and every one of its electrons is important since one of its laws is that electrons cannot be lost, not even one; they can change form, but they cannot be destroyed.

Big things are made up of a lot of little things. How important is one of the little things to the big thing? How important is one opinion out of billions? Or one vote out of millions?

As I survey the public sources of time in the town where I live, schools, radio and TV stations, and banks, I notice that none of them seem to give the same time. The total span of time they cover is about 10 minutes and this determines what ‘being on time’ means to clock conscious people. Being on time seems to be a personal thing and I know people who feel that being on time means getting there within a day or two of the appointment.

I have been at schools where all the clocks showed different times and I was never quite sure what being on time to class meant. I once gave out a teacher evaluation that asked if I was on time to class. Half of the class said that I was always on time and the other half said I was never on time. I took this to mean that half of the class was there before I was and the other half after. It had nothing to do with the actual time on the clock.

Seconds are the smallest units of time that I have much intuitive feel for and I consider them to be my personal basic unit of time. I use a “one-hippopotamus, two-hippopotamus...” approach if I am going to be serious about time in a casual way. In the time flow of my life, a difference of ten microseconds is not meaningful.

This is not to say that ten microseconds is never meaningful. My computer measures time in microseconds and can distinguish events that are microseconds apart.

I think that my intuitive feeling of time comes from several sources: the particular set of senses I come equipped with, the length of my life so far, the rhythm of my heart beat, and the motion of the earth around the sun. There are probably more but I easily recognize these. If I am waiting for someone, I don't need a clock or a calendar to tell me if I have been waiting seconds, minutes, hours, days, weeks, months, or years.

As two events occur closer and closer together in time, there comes a point when they seem to be simultaneous. There is a point at which I hear two guns being fired at the same time, even though they are not. There is a point at which I see two flashes of light as happening at the same time, even though they don’t. There is a length of time so small that it seems like an instant to me. These minimal perceptions of time depend on the physical construction of my body.

I can see a flicker in a light bulb that that uses 50 cycle/sec. electricity, so I have some perception of 0.02 seconds. On the other hand, if I watch a top fuel dragster do the quarter mile in under five seconds, I can intellectually understand the accomplishment but I have to look at the posted time to see that it was 4.98 seconds and not 5.03 seconds.

At the other end of the scale, it is my understanding that the dinosaurs were on earth for over one hundred million years. This amount of time staggers my imagination. I have been told that my species has been here for about three million years and I can not intuit three million years. What would it feel like to look back at a hundred million years of procreating the species and eating tree leaves? Perhaps the extinction of the dinosaurs was not the result of cosmic catastrophe but was a mass suicide brought on by boredom. I suppose it is human to think that boredom is worse than death.

I find it hard to consider, even intellectually, an interval of time longer than the age of the universe.

So, I do not intuitively feel very short times or very long times. I think that the longest time that I, personally, have any intuition for is the length of time I have been alive. Perhaps part of the concept of generation gap is that a person who has been alive ten years can not fully grasp what fifty years means.

Because of the society in which I live, my time sense is artificially extended by technology. There are clocks that can measure nanoseconds. Electromagnetic radiation, for example light, travels about three billion meters in ten seconds, so it takes about three and a third nanoseconds for it to travel a meter. When I talk about nanoseconds, I am talking about the length time it takes electricity to run down a not very long wire. My extended senses see intervals of time this small.

The technology of measuring instruments allows me to make statements about the amount of time between physical events that I see or sense as simultaneous. I stand at one end of a room and a lamp hangs on the opposite wall. I have a gun in one hand and the cord of the lamp in the other. If I simultaneously shoot the gun at the opposite wall and plug in the lamp, it will seem to my senses that the bullet and the electricity get across the room in a dead heat. But with my extended senses, with my very accurate clocks, I can see that electricity is the clear winner.

There are several ways my senses are extended to long lengths of time; history gives me a sense of centuries, archaeology a sense of millennia, paleontology a sense of millions of years, geology a sense of millions and billions of years, and astronomy takes me all the way back to the end of whatever the universe was before it became our universe, more or less.

I want to get back to the rock.

LECTURE 1-3

So I frequented hospitals and stables, noting exact times of birth, hoping that some of the babies would become jockeys and some would become race horses...

I feel that my description of the motion of the falling rock is at a dead end without more data, in particular, the times it takes the rock to fall its appointed distances. This remark highlights the narrowness of my view since I could talk about the way the rock looked silhouetted against the oleanders that line the alley behind the apartments, or how, momentarily, the sun reflected off a line of quartz in the rock as it plummeted to earth. I could, but I don't. My bent is toward numerical descriptions and I am going to hold that line. At heart I am still a boy throwing firecrackers into the river.

Well, I have a basic unit of time, the second, that I feel pretty good about, and I can put how long it takes the rock to fall in my description. I am going to repeat the experiment a lot of times using the "one-hippopotamus..." method to count the seconds it takes the rock to fall from the three different heights.

Having done these experiments, it seems to me that the most striking numerical aspect of dropping rocks is the relation between the distance the rock falls and the time it takes. As near as I can count seconds, a rock dropped from the same height always takes the same time to fall. To each height there is a time and there are different times for different heights. Actually there was a little bit of variation in the times from a particular height, but considering the clock I used, this is hardly surprising.

I didn't measure the heights, but I did hold the rock as close as I could to the same place each time I dropped it. I feel quite sure that the rock fell very close to the same distance each time I dropped it at a particular site and that for each height there was one time that it took to fall.

Well, this doesn't just happen all the time. It doesn't happen, for example, if I drop a feather from my fingers, a leaf from a tree or a Boeing 707 from a couple of miles. But if I am dropping rocks, I have indications that there is a close relationship between distance and the time it takes a rock to fall that distance. It would be interesting if such a relation did exist and I think that I will check it out a little more closely. To that end, I will measure the distances and get a better clock.

LECTURE 1-4

While I waited for them to grow up, I looked for the rule that would relate their ages to financial success. I was sure there was one...

Time has always seemed a mysterious, elusive quantity to me whereas distance has always seemed kind of warm and “homey.” I suppose this is because I die at the end of some interval of time but not at the end of some interval of distance.

Time often dominates distance. I think of Tucson as being eight hours away from Albuquerque and not four hundred miles. When I think that the trip will never end, I’m not thinking about the distance, I’m thinking that the time will never come when this trip is over.

But distance is an equal partner with time in the description of motion, and it deserves some consideration, even if it is not quite as dashing as its compadre. So how do I feel about distance?

First of all, I am going to measure distance in the usual way using fractions of inches, inches, feet, yards, miles, and light-years; or, millimeters, centimeters, meters, kilometers, and light-years. I grew up using inches, feet, yards and miles so I tend to think in this system and translate to metric, contrary to the way most of the world does it. Also due to an accident of birth, I measure some distances in terms of football fields.

Because I am equipped with a set of senses that are particular to human beings, I can feel some distances intuitively and others not. When I look into a tool box I find it hard to distinguish between a 7/16 in. and a 1/2 in. open end wrench but I suppose that there are mechanics who can.

It is possible to buy a ruler that is marked in sixty-fourths of an inch. If I am doing work around the house, my pencil line is wider than 1/64 in. but with a sharp #2 pencil, I can work with 1/32 in.. The smallest distance between marks on a carpenter's tape measure is 1/16 in. and I am pleased when I mark and saw a board within 1/16 in. of what I want. A good machinist talks about a thousandth of an inch.

On the metric side, I can feel the millimeter intuitively and there are rulers that are marked in millimeters. While I am not sure that I can tell a 17 mm. socket from an 18 mm. socket, I can tell 17 mm. socket from a 14 mm. socket.

The smallest distance I can see is determined by the construction of my eye. The tips of my fingers can feel a smaller distance than my eyes can see, and my tongue does a better job yet. I find it hard to think about something as small as an atom.

The unit of measurement that I actually do use depends on the size of the thing I am measuring and on how accurate I have to be. Meters and feet work pretty well until I get to distances as large as a kilometer or a mile.

Heights of people and dimensions of house lots are given in terms of feet or meters. I think inches work well for the length of a trout or a baby. Atomic distances are given in terms of meters and the magnitude is around 10^{-10} meters.

Between the millimeter and the kilometer there are two intermediate steps, the centimeter and the meter. There are no steps between the millimeter and 10^{-10} meters and there are no steps between a kilometer and a light-year.

I measure the distance from my house to the street corner in feet and the distance from my house to Santa Fe in miles. I measure the distance from my house to Santa Fe in the same units that I use to measure the distance from my house to the sun. The largest distance that I can think of is the diameter of the universe, although I am not quite sure what this means

When I dropped the rock from my hand to the ground, it fell 5 feet, 1 9/16 inches according to a standard tape measure. I love fractions, feet and inches dearly but this is not the place for them. The rock fell 5.11 feet. Well, it fell <u>about</u> 5.11 feet.

To measure the distance the rock fell when dropped from the apartment building, I used a 35 foot cord to make a mark on the apartment building that was 35 feet from the ground. Considering that the cord stretched some when it was held taut, this measurement was good to about 1 inch.

I did not get a good measurement for the depth of the hole so I am going to drop this case. I tied a weight to a cord and let it down the hole but I could not really tell when it hit the bottom.

I measured the time for the rock to fall from the apartment building as 1.5 seconds and to fall from my hand as 0.5 seconds. The fractions are not very accurate. It seemed to me that I was about half way through "two-hippopotamus" when the rock hit the ground from the top of the apartment building. When I dropped the rock from my hand to the ground, I was about half the way through "one-hippopotamus" when it hit.

This is cheating a little. The ‘hippopotamus’ method is a method for counting seconds and to break the words up into fractions of a second is not justified. I can feel the length of a second so I just ‘know’ it has parts and I am tempted to estimate parts of a second. I could see that the rock hit somewhere in the middle of a second and I felt compelled to estimate where. But if I say that I can measure seconds, then it is seconds that I should measure. The rock took 2 seconds to fall from the apartment building and 1 second to fall from my hand.

I don’t have this compulsion to break up very small intervals of time because they seem like instants of time to me, whereas the second seems to have a definite duration. Events that are 1/1000 second apart seem simultaneous to me. If I read 3.456 seconds on the digital readout of a clock, I don't think of breaking up the interval of time between 3.456 and 3.457. I can't feel the interval of time between 3.456 and 3.457 seconds so, to me, that interval is zero. I think, "Let's see, 3.456 seconds. Right on."

It seems needlessly rustic to measure time in ‘hippopotami’ when the market abounds with stop watches that are good to a hundredth of a second, so I will use one of them.

The stop watch worked pretty well for the apartment building. I had a friend drop the rock and I could see when the rock was dropped and when it hit the ground. We did it several times and the times ranged between 1.52 and 1.57 seconds. The variation could have been due to variations in my reaction times. Also, the placement of the rock was only good to 1/16 inch and these differences would give different times.

The range was still pretty narrow when I dropped the rock from my hand; between .58 and .63 seconds. These results certainly did not contradict my feeling that for each height there was a single time it took the rock to fall. I am personally convinced that it always took the same time for the rock to fall from the apartment building and likewise when I dropped the rock from my hand.

So where do I want to go with my conviction? If I am correct, then I see no reason why these two heights are special and it is probably true for all heights...most heights...a lot of heights. I think that I will see what the times are for several different heights.

If for each height there is, indeed, a unique time it takes the rock to fall, then there is a possibility of measuring height as the time it takes a rock to fall that far. I could say that the apartment building was 2 seconds high, meaning that the rock would fall that far in 2 seconds.

While the method of measuring distance as the time it takes a rock to fall that distance has a certain charm, it has a severe drawback. There is no basic unit of distance. I would like to say that the basic unit of distance is the distance that a rock falls in one second. This is done in astronomy with the light-year; the distance that light travels in one year. But in that case light travels the same distance in any year, and the measure of distance does not depend on when light traveled the distance but only on the length of time it took. Light travels twice the distance in any two years that it travels in any one year.

I can look at the falling rock and see that it doesn't fall as far in the first second as it does in the next second. I can see that a rock falls considerably farther in two seconds than twice the distance it falls in the first second. I can see this for the first two seconds anyway. I rather suspect that the rock will fall different distances during any two different one second intervals of time. Because of this non-uniformity, I think that it would be very difficult to measure lengths this way. For example, I don't at the moment see how to measure the horizontal distance of a mile.

LECTURE 1-5

But the births always seemed to occur between the ticks of my clock and the exact times eluded me. This must explain my continuing losses...

While it is true that I have only two experimental results, I do not live in the vacuum of these two experiments. I live in a world of motion and falling objects. It is seldom that I walk over a bridge and not drop something into the water below, watching it carefully all the way to the splash. I continue a habit that I began as a child of trying to drop pebbles on other pebbles while waiting for a bus. The two experiments only tend to verify what I already knew in my heart; given a rock, there is one time it takes to fall a given distance and one distance it falls in a given time. I believe that if I drop my rock, it always falls sixteen feet in the first second and that in falling the first sixteen feet from rest, it always takes one second.

Of course this belief is naive and possibly not true. Kepler's naive belief was that the orbits of the planets were circles. They were awfully close to circles and to the accuracy I've used so far with the rock, they were circles. Greater accuracy of measurement revealed the orbits to be ellipses. Perhaps greater accuracy will show my belief to be in error also.

It seems to me that if there is a unique time that it takes a rock to fall a given distance, then if I am given a distance, that unique time is determined whether I drop the rock or not. Holding steadily to my belief, I wonder if there is a way to find that time without dropping the rock. This is where my interest lies. Had I been a bird breaking clams on the rocks below or a poet, it is possible that some other essence of the rock would have caught my attention.

My approach is experimental and the idea is to drop the rock from a lot of different heights, measure the times that it takes the rock to fall, and see if I can discern some pattern or relationship between distance and time. If I am going to make precise statements about distance and time, I am going to have to get down off of apartment building roofs and into a laboratory. The ground is uneven at the place where I drop the rock, it is hard to drop it from exactly the same place each time, and my reaction times are not all that great when it comes to stopping and starting the watch. These are problems that can be dealt with to some degree, if not completely, in a laboratory.

I am thinking about dropping the rock one foot and then two feet and then three feet and so on up to, say, sixteen feet. It occurs to me that if the rock falls sixteen feet, it also falls all the distances between zero and sixteen feet in the process. I will try to remember that. It might come in handy some day.

There is one point that I would like clear up before going on. I have been dropping the same rock over and over. I dropped it sixteen feet and according to my clock it took 0.98 seconds, which I'm going to round off to 1.0 second. If I dropped another rock, say a heavier rock, would it take less time? Do I have to keep this rock with me during the entire experiment? Does every rock require its own experiment?

It would be nice if the objects that ‘fall like rocks’ all fall sixteen feet in one second or short of that, if there were some simple way that time related to the weight of the object, like, if a rock has twice the weight of another then it falls in half the time.

In the late 1500's, Galileo, on the basis of experimental evidence, decided that objects that fall like rocks all take the same time to fall a given distance. He claimed that a 10 pound rock and a 20 pound rock would both take the same time to fall from any convenient, near by, leaning tower. This experiment has been done several times since Galileo with increasing degrees of accuracy, and the results always agree with Galileo. I am willing to accept it. This is one of the basic assumptions of physics and it was arrived at experimentally, not theoretically. Someday it may be determined that it takes heavier things just a little less time to fall, something out in the twentieth place maybe, but for now, I am going to assume that all ‘rock like’ objects fall a particular distance in the same time. I can retire my faithful rock.

This is a little counter-intuitive to me. I would not have been surprised if heavier objects took less time to fall. Aristotle thought that heavier objects took less time to fall and he was supposed to be a pretty bright guy. I guess that even pretty bright guys can be wrong sometimes.

I plan to do some experiments which I hope will give me insight into the relation between the distance a rock falls and the time it takes. Since I plan to represent time and distance with numbers I am first going consider what it means to represent these quantities with numbers.

LECTURE 1-6

In a just universe my efforts would have been rewarded. Alas...

One of these days I am going into a laboratory and come out with a relation between the distance a rock falls from rest and how much time it takes. I am going to use finite decimals to represent time and distance and I am going to get these numbers by measuring time with clocks and distance with rulers. I am then going to try to find, by hook or by crook, a relation between these numbers.

Since the way I experience the real world is through my senses, the natural five along with rulers and clocks, my perception of the real world is determined by what these senses tell me. I don't know what the real world is, I only know what I perceive it to be, and I take that perception for reality. As my perception changes, my reality changes.

First, I want to consider what the numbers I read from clocks mean.

The critical characteristic of a clock is the length of the smallest time interval that it can measure and this affects every value of time measured by the clock. If it can measure 1/1000 second, then it can distinguish between 2.345 seconds and 2.346 seconds and the times on this clock are good to three decimal places. If this clock read 2.345 seconds, then the first four digits of any better clock would read 2.345... . Whatever the ‘actual’ time is, and I allow the possibility that the ‘actual’ time can not be described by a single number, I feel as sure as I feel about anything, that the first four digits of any decimal representation of that time will be 2.345... . I guess that I am assuming the watch maker’s claims are correct.

However long it might have taken for the rock to fall from the roof of the apartment building, I know that the first digit of the decimal that represents that time in seconds is 1. It's the rest of the digits that I'm not sure about.

In any given situation, there is a smallest interval of time that I can measure in the sense that there is a positive integer, n, where I can measure 10^{-n} seconds but not $10^{-(n+1)}$ seconds; for example, perhaps I have a clock that measures 1/100 second but not 1/1000 second. Time, as measured by the clock, goes from 3.46 seconds to 3.47 seconds; there is nothing in between.

Without a clock, I think that on a good day I could measure seconds but not a tenth, so my personal n equals zero. The value of n could be larger than 9 in a sophisticated laboratory.

Suppose that 10^{-n} seconds is the smallest interval of time that I can measure. Then my perception of the real world is that there are 10^{n} instants of time in every second. I think of taking a second and breaking it into 10^{n} equal parts or 10^{n} instants of time. These instants are the constituent parts or building blocks of the second. If n equals one, then every second is broken into ten equal parts. The length of time that the rock falls is the sum of the lengths of all the instants of time that occurred in the descent. An instant of time is a positive amount of time.

If I start my clock at zero when I get to the bus stop and stop it when the bus comes, at 2 minutes and 34.5 seconds, the time I read on the clock is the amount of time I had to wait for the bus. This is the sum of 1,545 instants of time that are 1/10 second long. If I look at the clock at an intermediate time, say it reads 1 minute, then I have been waiting for six hundred instants of time. The '1 minute' represents the length of an interval of time.

But there is another way to look at the one minute reading. It is a point in time that locates me between my arrival at the bus stop and the arrival of the bus. If I have an appointment at 3:30 in the afternoon I don't **think** of it as being three and a half hours after noon, although that is certainly true. I think of 3:30 as locating the point of time in the day when I have to be in court.

I can see the instants of time as beads on a straight string, although I don't see any real necessity for it being straight. The beads successively light up and stay lit as time passes. The number of beads that are lit is the number of instants of time that have passed and this gives me the amount of time that has passed. Each bead that lights is the head of a trail of lit beads which displays the amount of time that has passed.

If each bead goes out as its time passes, my focus is on the instant of time as it occurs. Now the lit bead is an instant or point in time. I can think that fifty instants of time have passed or I can think that I am at the fiftieth instant.

I can think of an instant of time as representing the amount of time that has passed up until then or as representing just that spot in time. An 'amount' and a 'spot' are not the same thing but I think of the instant in either or both ways as is appropriate to the situation. This type of double-think, and often multi-think, is typical in the mathematical description of physical events.

It is my **perception** that the number of instants of time in a second is finite. Since this is what I perceive, this is my reality.

If the smallest interval of time that my clock can measure is 1\10 second, then the reality is that there are 10 instants of time in any second. The clock ticks from 1.3 seconds to 1.4 seconds; there is nothing in between. When the clock stops, it stops on a number with a single digit after the decimal. These are the only numbers available to the clock. If I get a more accurate clock whose least measurable time is 1/100 second, my perception changes. It seems that my vision was a little blurred and that there are really 100 instants of time in a second. My reality has changed.

While I think of time as moving, I think of distance as stationary. The interval of time from sunrise to sunset is here and then gone. That exact interval of time will never happen again. The interval of distance between my front door and the street is there day after day and I can measure it at my leisure.

It won't be there forever though. I thought that I had plenty of time to measure the distance from my house to the street so I kept putting it off. Last May I drove to Tucson to visit my mother for a couple of days and when I got back I saw that the city had come in and widened the street. Now I will never know.

As with time, I have trouble measuring distances if they are very small or very large. If I am told that the treasure is buried half way between the two big oak trees in the front yard, I have no trouble figuring out where to start digging. I am less sure about finding a point half way between the two hydrogen atoms in a molecule of water.

There is a smallest interval of distance I can measure in the sense that there is a positive integer n where I can measure 10^{-n} meters but not $10^{-(n+1)}$ meters. I think my everyday, unaided n is 3. I can see a millimeter but not a tenth of a millimeter.

Atomic distances are of the order of 10^{-10} meters and I suppose this is known because they have been measured.

Regardless of how small the smallest distance I can measure is, it is always positive and as far as I can perceive, there are a finite number of points on the line between any two points. If 0.1 centimeters is the smallest distance I can measure, then there are 10 points in any centimeter. Since this is my perception, this is my reality.

Here, also, I am thinking about cutting the centimeter into ten equal pieces and the lengths of the pieces add up to the length of the whole. This seems clearer to me than the decomposition of the interval of time. A centimeter is easier for me to picture than a second. Time, the ultimate commodity; I know what it is until I think about what it is.

I can think of the points on the path of the rock as beads on a straight string and straight is appropriate here. A point bead lights up and stays lit when the rock gets to it. When the twenty-third bead lights, the rock will have fallen 2.3 centimeters. The string of lit beads represents the amount of distance the rock has fallen.

I can think of the point bead lighting when the rock gets there and going out as it leaves. Now my attention is on the single bead, which represents the spot where the rock is. As the bead lights I can think of it as representing the amount of distance traveled or as the spot on the path that is the present location of the rock.

I notice that the point beads need the rock to make them light. Something else seems to make the time beads light.

LECTURE 1-7

I have given up betting on horses and joined a Buddhist monastery. Mathematics is not for me...

No matter what kind of laboratory I am in, there is a smallest time and a smallest distance I can measure. This means that as far as my perception of the physical world goes, there are a finite number of instants of time between any two times and a finite number of points on a line segment between any two points in space. How many instants of time I put in a second is a decision that I make in the laboratory and depends on the equipment I have available and on the context in which the number is going to be used.

The measuring instruments and context give me a perception of reality that I take to be reality.

The rock starts at exactly one of these instants of time, the zero instant, and at exactly one of these points in space, the zero point. It ends up at exactly one instant of time and one point in space. The laboratory problem is to find out what these instants and points are and this is not so easy.

Besides the 'smallest measurable size' limitations inherent in a clock or ruler, there are a variety of errors that can enter into measurement. There are errors in starting and stopping the clock, for instance.

The starting of the clock approximates the beginning of the event and the time on the clock when it is stopped, approximates the end of the event.

If my clock measures 0.01 seconds and if my reaction time is two hundredths of a second, then a clock time of 0.73 seconds means that the actual time lies between 0.69 seconds and 0.77 seconds, assuming that the reaction time was the only error. The clock time could have been the result of starting it 0.02 seconds late and stopping it 0.02 seconds early, in which case the actual time would be 0.77 seconds. But it could also have been the result of starting the clock 0.02 seconds early and stopping it 0.02 seconds late, in which case the actual time would have been 0.69 seconds. What really happened is somewhere between these two extremes.

The clock can measure 0.01 seconds and so there are 100 instants in a second. There is some actual number of instants that the rock falls but all I know for sure is that it is one of the numbers {69,70,71,72,73,74,75,76,77}. That is as good as I can do with this dime store watch and my thumb. I can get a better watch and micro-switches, but I can never tie down the number of instants that the rock falls exactly.

The point is that laboratories can not give an exact number.

I would like to say that no physical quantities can be measured exactly but hyperbole is always dangerous and I guess this is no exception. The earth has one moon, I have two parents, and Orion's belt has three stars. Physical quantities that can be expressed as small positive integers can be expressed exactly. No others can. Laboratories do not give exact answers, they give answers with less error than doing it at home.

There was a time when I thought that quantities that could be expressed in terms of large positive integers were also exact, but no longer. When I read that 101,435 people attended the Notre Dame-Michigan football game, I don't believe it. I do believe, however, that over 100,000 people attended the game, whatever 100,000 means.

An ordinary watch ticks in seconds. There is one instant of time in a second. The second hand moves around the dial in sixty discrete, forceful steps. It seems to instantaneously go from one ‘second’ mark to the next. There are sixty instants of time in a minute. I really like that. Here is an instant of time that I can really get hold of.

When I was first dropping the rock and measuring how long it took with my dime store watch, I knew what the numbers meant. There were a hundred instants to the second and it was more like grains of sand on a string than beads but I still had a good feeling of what was going on.

Commander Cody sings about a line of telephone poles turning into a picket fence as he speeds by in his "Hot Rod Lincoln". I'm going to ‘turn up the wick’ and make it into a wall. I go to my local scientific supply house and get a clock that measures 1/1000 second. There are no beads now, just a barely bumpy string. My physical being can no longer think of each instant separately, just as it can no longer distinguish separate telephone poles.

I can understand that a rock could fall a foot in a quarter of a second, four feet in a half of a second, sixteen feet in a second, whatever. These are all numbers that I can understand intuitively and I feel good about them. I can even see a possible real world pattern emerging. My intuition begins to lessen when I go to the laboratory and find that the rock falls 0.99999856734 feet in .24999994736 seconds.

I have little feeling for the numbers in the eleventh place and as accuracy grows my intuition fades until it finally fades away in a sea of tinyness.

My intuition is engendered by sight and feel and this kind of accuracy is significant only in places that are far beyond sight and feel. This is a place where only my intellect can go and my intellect must eventually face the necessity of developing its own intuition. I must pass from physical intuition to intellectual intuition and it is my opinion that one of the biggest steps that must be taken in walking to the world of science and mathematics is moving from physical intuition to intellectual intuition.

LECTURE 1-8

I realized that justice was not in the job description of the universe...

Since I can't experience very small intervals of time or distance directly or with my instruments, the very small universe is hidden from me.

One possibility for the 'very small' universe is that it is discrete. At some point, space and time just can't be subdivided any more. There is nothing in between two instants of time.

Another possibility is that at some point the numbers become random. For example, no matter how accurate the clock is, the number in the hundredths place, say, would show no preference for one digit over another.

The truth is, I don't know what the very small universe is like, or the very large for that matter. There is a wall, beyond which I can not see. On my side of the wall is that part of the real world that I can sense with my measuring instruments. On my side, I can do experiments which give me insight into how these things that I sense relate to each other. My view of the wall, however, is not unobstructed and the closer something is to the wall, the harder it is to see. Away from the wall I can use meter sticks, scales and clocks to measure. Near the wall I use electron microscopes and linear accelerators to see the small; telescopes and 'very large arrays' to see the large.

I can only speculate about the other side of the wall. I have no idea what a time interval of
10^{-100} seconds or a distance of 10^{-100} meters might mean, but if there is an activity that humans do, long and hard, it's speculation about the other side of their own particular walls and I share that trait with the rest of my species.

So here I am, about to slide into the metaphysical, with a real world that is often self evident but surprisingly slippery when it comes to tying it down with numbers. The way out of this morass is the idea of a model. The real world is just too hard to take as it is and I want to take the part I'm interested in and make a model of it. If I make the model correctly, what happens in the model should happen in the real world, more or less.

That's a pretty amazing idea. I guess the catch is that the model must be made 'correctly' without knowing quite what 'correctly' means. I'm going to ignore this for the moment and consider what general attributes I would like the model to have.

First, the model should be more understandable than the piece of the real world it models. The model should agree as much as possible with known experimental knowledge. The real world is a tough place to measure and measurement in the model should be easy and exact. When I say that a rock in the model falls 9 feet in 0.75 seconds, that's exactly what it does, no 'plus or minus'. Distance and time are measured with instruments of infinite precision and these exist in the model.

In the real world it seems virtually impossible to get exactly the same response from a given initial condition, partially because it is virtually impossible to repeat the initial condition. In a model I want repeatability. I want to hold that rock 9 feet from the floor whenever the spirit moves me and whenever I drop it, I want it to take 0.75 seconds to hit the floor.

I was talking once to a person who had been active in the design of a very large, successful airplane and I remarked that the airfoil must have required a lot of engineering. He replied that actually the design had just been ‘eye balled’ and made into a model for the wind tunnel. They adjusted the model until it flew well. I suppose that there was some exaggeration in the story, but it points up how important the model is. It is often so much easier to work with a model than with reality.

The purpose of the model is to help me understand things that I don't understand. I see a rock fall and it is part of my observable universe in that I can measure where the rock is at various times. I don't, however, understand why it falls and I don't understand how it falls. If I can understand these things in the world of a model, then perhaps I can understand them in the real world. The model gives me insight into how the real world works on my side of the wall.

My knowledge of my side of the wall comes from experimentation. The closer the fit between the model and my knowledge, the more confidence I have that the model is giving me good information about the gaps in my knowledge. On the other hand, since the model is simpler than reality, there is always some doubt that the model accurately fills in the gaps. This means that the insight offered by the model must always be backed up by experimentation. A model can not give truth about the real world, it can only indicate the direction of truth. Real world truth can only be established by experiment.

Since experimental technology keeps changing, so does real world truth. As measuring instruments got better, the ‘truth’ about disease changed, the ‘truth’ about the age, shape, and motion of the earth changed, the ‘truth’ about age, shape, and motion of the universe changed. As real world truth changes, so must the model.

There is another use of models that is both exciting and dangerous, and that is filling in blanks on the other side of the wall. It is exciting because it seems to give knowledge about the unknowable. I often know what the model is like on the far side of the wall so I can at least consider the possibility that reality is something like the model over there.

I would like a model that fills in the blank on the other side of the wall of measurement. What is the small universe really like? Theology looks for models that fill in the blank on the other side of the wall of death. Evolution and Genesis are models that fill in the blank on the other side of the wall of ‘where did I come from?’.

I need to experiment to verify what the model tells me about my side, but I can't experiment on the unknown side; if I could, then it wouldn't be the unknown side of the wall. So I don't have a way of knowing how much, if any, reality the model reflects of the unknown side of the wall. When physicists include the big bang in their model, they are on the other side of the wall and it is hard for me to imagine the experiment that would verify any particular version of the origin of the universe. A physicist might disagree with that remark.

The danger arises from the fact that there can be two or more models that fit the ‘known’ side of the wall very well and are very different on the other side.

The danger becomes concrete when people think that their interpretation of the other side of the wall <u>is</u> reality. It becomes psychotic when their perception of the real world is altered to accommodate a model that would coincide with a preconceived notion of the far side of the wall. Since the proponents of various interpretations of the far side of the wall can't experiment to settle their differences, the argument is often carried out with fire and sword.

I think of many of my favorite models as living in a fantasy land like ‘Oz’. They are in an ideal world where if I do ‘this’, then ‘that’ will always happen. Always. If I strike on the tinder box three times, a dog with eyes as big as dinner plates will always appear. There are a finite number of laws in this land and they are fiercely enforced. Truth is legislated to be a property of that which obeys the laws. The ideal geometric figures of Plato are found in this land as is mathematics in general. All of those juicy numbers like ‘e’ and ‘π’ are here.

For no particular reason, it looks like the Midwestern United States in my mind's eye; mostly rural with few small cities. Rocks fall like rocks, toasters heat up, airplanes fly, stereos play, things just don't look all that much different. But there are some very strange concepts in this land, like infinity. If a version of Iowa is in the fantasy world, is a version of infinity in mine? Nah.

So the question is, "What happens when I strike the tinder box in the real world?" How much of what happens in the world of the model happens in the real world? Some models are very good and their predictions have been verified by experiment many, many times. How good a track record should a model have on verifiable predictions if it is to be given credence on non-verifiable predictions?

LECTURE 1-9

...and from my mountain retreat I watched the great strides made in understanding the rules that predicted the future if the present were known...

I am going to consider two worlds, a Real World and an Ideal World. Problems arise in the Real World and are modeled in the Ideal World. In particular, I am going to use mathematics to model Real World problems in the Ideal World.

The problem is solved in the Ideal World using mathematics and the process of solution is driven by mathematical intuition with maybe a few touchstones of reality along the way. The Ideal World solution is now interpreted in the Real World. This interpretation is the model's prediction of what is going to happen in the Real World. Then an experiment is performed to see if the prediction is correct.

The problem was the motion of planets around a sun. Newton modeled his laws of nature in mathematics and used concepts of mathematics to solve the problem. His solution in the Ideal World was that the paths of the planets were ideal ellipses. These could be interpreted in the Real World as Real World ellipses. Providence, Tycho Brahe, and Kepler put together an experiment, and sure enough, the paths of the planets in the experiment were Real World ellipses. The actual order of events was a little different but. this is the general idea.

The characteristics of these two worlds will become manifest as I continue, but I want to make some general remarks about them now.

The Real World is where the action is. It is where I live. I am not even sure the Ideal World exists; sometimes I think it does but most of the time I think it doesn't. If the Ideal World does exist, I wonder about how it might have come into existence. Was it invented or discovered? Or something else?

In the Real World there is only a finite amount of numbers, a fairly small subset of the finite decimals, that is, decimals with a finite number of places. The physical quantities of the Real World are approximated by these numbers.

I am limited to finite decimals because that is all I can write in the Real World, and they are used to approximate physical quantities because they are the kind of numbers that Real World measuring instruments return. There are a finite number of points in space and a finite number of instants of time and the number of each depends on the smallest amount of each I can measure.

The Ideal World is, of course, ideal. The Ideal World is the fantasy land where I want to build my models of the Real World. I want mathematics to be part of the Ideal World and I also want mathematics to be the language that I use to express Ideal World ideas in the Real World. So mathematics must not only be itself, the ideas and concepts, but must also be the language that expresses itself in the Real World.

If I write music, I use standard musical symbols and a pretty well-defined language. I can give the score to a musician who makes the music. There is no confusion between the language and the music.

With mathematics it is less clear. The symbols ‘2’ and ‘3’ stand for the concept of a certain number of objects in a set. As such they seem reasonable elements of a language, and the expression, $2 + 3 = 5$, represents in language the thought that 2 things and 3 things together is 5 things, in much the same way that musical notes on paper represent a tune in someone’s head.

On the other hand, the only way I can conceive of the mathematical idea,
$4567 + 9876 = 14443$, is to write it down just that way. The expression,
$4567 + 9876 = 14443$ <u>is</u> the thought in my head and I can’t think of it in any other way.

The symbols and expressions that represent concepts of mathematics are part of the concepts of mathematics. A written language like English or the notes and clefs of music are there to express ideas that already exist. The language of mathematics generates new ideas. Part of mathematics, the concept, is the manipulation of mathematics, the language, to get new mathematics

By the term ‘mathematics’ I will mean neither more nor less than I want it to mean on the occasion of usage. That means I am not going to define exactly what I mean by ‘mathematics’, but I am going to use the term as if I had, a common and useful technique in all walks of life.

Mathematics resides in the Ideal World and brings with it one of the key concepts of the Ideal World, infinity. There are an infinite number of points on a line, and an infinite number of points between any two points on a line. There are an infinite number of numbers and this includes an infinity of positive integers, of rational numbers, of irrational numbers, of prime numbers, on and on. There can be an infinite number of instants of time between any two instants of time.

The ideal geometry of Plato is in the Ideal World. The length, width and height of a point are zero, the height and width of a line are zero, the height of a plane is zero. Curves have zero width and height. All of these objects are composed of an infinite number of points.

Of course, I can't enter the Ideal World physically and it used to be something of a disappointment to me that I couldn't experience the Ideal World directly. It seemed so clean and pure. But it is also austere and unforgiving and I now am quite happy in the Real World, making visits to the Ideal World only in my mind.

Since I can't go to the Ideal World and since the objects of the Ideal World can't be brought into the Real World, I am stuck with describing the Ideal World in Real World terms. This is not easy because there are ideas in the Ideal World that do not have Real World counter parts and so there are no Real World analogies to help describe them. Infinity is an example of this. Mathematics provides a language written in Real World symbols that helps me with Ideal World infinity.

The symbol π represents a number in the Ideal World that does not exist in the Real World. The symbol 0.25 represents a number in the Ideal World and the Real World. The symbol ‘Jeff’ represents an object in the Real World that is not in the Ideal World.

I can't experience an ideal circle, so with a stick in the sand, or with a pencil on paper, or with a mouse in the graphics program, I draw a Real World circle. I draw a picture that helps me think about and, perhaps, to some degree understand the ideal circle. In the Real World I have the exquisite circle of the draftsperson, the elusive circle of Zen and the halo of sainthood, but a favorite poet, Edna St. Vincent Millay, says that, “Euclid alone has looked on beauty bare”.

The infinite, non-repeating decimal exists as a complete entity in the Ideal World. I don't really see what this object ‘looks like’. I don’t see it because I can't conceptualize an infinite, non-repeating string of digits in its entirety. I don’t even know what most of the digits are in an infinite, non-repeating decimal.

Infinity exists in the Ideal World but not in the Real World. I am able to believe that a singing, dancing scarecrow lives in the land of Oz and I am able to believe that infinity lives in the Ideal World. Infinity doesn't have to make any more sense in the Real World than The Scarecrow does. Infinity makes sense in the Ideal World and The Scarecrow makes sense in Oz.

Making Real World sense out of what goes on in the Ideal World is only part of the problem. I must also be able to express Real World ideas in the language of the Ideal World if I am going to model the problem there. This language is also mathematics.

For the sake of convenience I am going to put a sentient being in the Ideal World that understands the concepts and objects of that world and in particular understands infinity the way I understand 0.25. I will call this being ‘God’.

LECTURE 1-10

I have seen men on the moon and the big red eye of Jupiter from the vantage point of a space probe...

Two numbers in the Real World are equal if I can't measure any difference in them. If 1/1000th inch is too small for me to see, then 0.456 in. and 0.457 in. measure the same and, as far as the Real World is concerned, they are equal. If I have an infinite, non-repeating decimal that begins 0.45612..., then as far as I, in the Real World, am concerned, it equals 0.456 because I can't measure the difference between 0.456 and the infinite, non-repeating decimal. Equality in the Real World is a matter of measurement.

In the Ideal World, a number is equal to an infinite, non-repeating decimal if it, too, is an infinite, non-repeating decimal and the numbers in each place are exactly the same. If two numbers were the same decimal except in the fiftieth place, they would be equal in the Real World and unequal in the Ideal World. In the Ideal World equality means "exactly the same as".

This means to me that the infinite, non-repeating decimal must, in some sense, be aware of the entire infinity of numbers in its expansion. If they are going to tell themselves apart from other numbers, they must be aware of their value in every place. God knows how they do it.

When I was quite young my father pointed out to me that no matter how big a number was, I could always add one and make it even bigger. I can also recall the frustration I felt as the realization dawned on me that I would never be able to comprehend the infinity of counting numbers in their entirety.

In the Ideal World the ratio of the circumference of a circle to its diameter is a constant and this constant is an infinite, non-repeating decimal . Since I don't know what the numbers are in the decimal expansion and since I couldn't write them down if I did, I have problems talking about it in the Real World. As I can't write it down like I do the number 0.25, I give it a name, π, so that I can talk about it.

It is very awkward talking about 'infinite, non-repeating decimals' so they are given a name. Such numbers are called **irrational numbers**. Numbers that are finite decimals or infinite repeating decimals are called **rational numbers**. The ubiquitous fractions are actually the rational numbers. Every fraction has an infinite repeating decimal representation and some fractions have a finite decimal representation. This dual representation arises from situations like 1/4 = 0.25000... = 0.24999... . The digit 0 repeats in the first representation and gives the finite decimal and the digit 9 repeats in the second case and the decimal is not finite.

I am reminded of the Ents in Tolkien's Lord of the Rings, whose names were their entire history. In a sense, 0.25's name tells all. Telling all about an irrational number is tough in the Real World. If I had to know all about a person to name them, I wouldn't even know what to call myself, so people have names like Ed and Mary. In the same way, I give names to irrational numbers that I run into a lot, like π and $\sqrt{2}$.

Like any other place the Ideal World has its ‘Badlands’. Here is where the more pathological aspects of the Real World are modeled, like the experiments that don't seem to repeat. And like most Badlands they are quite deep and beautiful when looked at carefully.

LECTURE 1-11

...but I haven't seen anybody beating the ponies...

I want to make some general comments about numbers. The traditional approach to mathematics is linear and it seems to be an, as yet unproved, folk theorem that there is a linear ordering of the ideas of mathematics where every idea follows from the preceding idea and all ideas are contained in the ordering. In my opinion such an ordering does not exist and I am not going to try to find one for the things I want to say about numbers. I think ideas clump into bundles of ideas that should be entertained simultaneously.

Unfortunately my senses are linear in time. I read one word at a time, I hear one word at a time, I speak one word at a time. I deal with this by loading the ideas linearly into my head and then try to consider them simultaneously. So I am just going to talk about numbers.

There are very few numbers that I have truly internalized. When I look at one, two, three, or four objects, I see one, two, three, or four objects. When I look at five objects, I see four plus one objects or two plus three objects. I have tried to count a group of people that numbered about 100, and failed. I could not, in the allotted time, get two counts that were the same. I do think that if I had more time I could have counted them exactly. I don't think it is possible for me or anyone else to count a group of around 100,000 exactly.

When the national debt went over one trillion dollars a man on television told me that a trillion one dollar bills placed end to end would go around the earth almost four thousand times. He was trying to explain one incomprehensible concept in terms of another. My immediate feeling about the earth is that it is flat and stationary and I have to go to some effort to convince myself otherwise. I often face south at sundown and, as the sun disappears, I try to really feel that the sun is fixed and the earth is rotating, taking me down, away from the sun.

What are big numbers? The edge of the universe is supposedly about fifteen billion light-years or 10^{22} miles away and that is about as big as I can imagine. It would take about 10^{90} motes of dust to fill it. These numbers mean almost nothing to me. It's a lot of miles and it's a lot of motes of dust; that's about it.

The numbers I've been talking about are pretty big. Am I getting close to infinity yet? Not really. Davis and Hersh in their book, The Mathematical Experience, discuss representing big numbers and they present a representation scheme that I find interesting. A number, *a*, inside a triangle is equal to a^a, so that a 2 inside a triangle equals 4. A number, *a*, inside a square equals *a* inside of *a* triangles so that a 2 inside a square equals a 2 inside of 2 triangles which equals 256. Continuing in the same way, a 2 inside a pentagon would equal a 2 inside of 2 squares, which equals 256 inside of a square, which equals 256 inside of 256 triangles, which equals 256^{256} inside of 255 triangles. POW! This is a big number. I know of no way other than a 2 inside a pentagon to express this number. It is beyond The Pale, beyond my imagination, beyond the wall. A '2 inside a pentagon' is an example of a finite decimal that is not in the Real World. There are not that many of anything in the universe; there is nothing that hot; there is nothing that far. As far as the Real World is concerned, it is infinite. In the Ideal World, it is not particularly large.

Now I have some big numbers and I think that I am surely close to infinity now. Again I am disappointed. There are as many numbers before me as there was when I started. The trip to infinity is a long one indeed and in the Real World we seem to be kept at home by the nature of things. I can't think of a number bigger than the biggest number I can think of and a two inside a pentagon is bigger than I can think of.

Suppose I ask God to pick a number between zero and infinity. What is the probability that a number between zero and a '2 inside a pentagon' is chosen? The probability is zero. Any finite chunk, and the chunk between zero and a '2 inside a pentagon' is finite, is puny compared with infinity. In fact it is zero compared to infinity.

Getting close to zero is much like getting very large; the trip to zero is every bit as long as the trip to infinity. As the positive integers get bigger and bigger, getting close to infinity, their reciprocals are getting close to zero. The positive integers never get to infinity and their reciprocals never get to zero.

Fractions are a mixed group. They are all in the Ideal World and a few are also in the Real World. 1/2 = 0.5 is in the Real World, 1/3 = .333... is not. Even though 1/3 is not a Real World number, it can represent a Real World process. I can sometimes divide something into three equal parts; for example, I can divide six dimes into three equal parts. I can not, however, divide a Real World pizza into three equal parts.

If I think of 1/3 as a symbol representing a process of dividing something into three equal pieces and taking one of them, as in the phrase, "1/3 of a pizza", a process that I am but rarely able to do in the Real World, then 1/10000 would mean that I divide something up into 10,000 equal parts. I don't think that there is anything that anybody can break into 10,000 equal parts in the Real World. I suppose God could do it in the Ideal World.

The decimal, 0.0001, is a Real World number but 1/10,000 does not represent a Real World process.

As a child I was told that 1/3 x 21 = 21/3 = 21 x 1/3 and that this was obvious. As an adult, I find it far from obvious.

I look at $1/3 \times 21$ as breaking 21 into three equal parts and taking one of them. So I break 21 into 3 groups of 7 and $1/3 \times 21 = 7$. I have $21 and I want to hire three kids to sweep out my bowling alley. How much do I pay each of them? Seven dollars.

Now 21/3 means division and division means repeated subtraction. 21/3 equals how many times I can subtract 3 from 21 and that is seven times. So 21/3 = 7. In this case, I have seven groups of three. I have $21 and I'm going to hire kids at $3 each to sweep out my bowling alley. How many kids can I hire? Seven kids.

The intuitions behind the two problems are quite different yet the answers are the same. Why should the size of one of three equal parts be the same as the number of times I can subtract three? Because in the first case I have three groups of seven and in the second case I have seven groups of three and $3 \times 7 = 7 \times 3$. I get the same number because multiplication commutes.

And what if I take the three all the way to the right and look at $21 \times 1/3$? Multiplication by a positive integer is 'fast addition', like $5 \times 3 = 3+3+3+3+3$, so $21 \times 1/3$ is 1/3 added up 21 times. If I add up 1/3 three times I get 1, and there are 7 'three times' in 21 times so $21 \times 1/3 = 7$. Some friends and I went to the mountains to talk and we brought some pizzas that we had cut into three equal parts. At the end of the afternoon we collected the left over pizza and found that we had 21 pieces. How many whole pizzas was that? Seven pizzas. It is amazing that mathematics has a single idea, fractions, that solves three such different problems.

Adding fractions is thought of as fairly elementary but I can recall assigning fractional partial credit to test problems and finding it impossible to add up the scores. I finally converted everything to decimals.

Another problem with fractions is that every fraction has an infinite number of symbols that represent it. Indeed,
1/2 = 2/4 = 3/6 = and so on. Do I have to deal with 26868/62692 or does it have some more tractable name like 3/7?

In the Ideal World, between any two numbers there is another number. This is not true in the Real World. If 1/1000 inch is the smallest I can see, then there is no number between 0.345 inch and 0.346 inch.

In the Ideal World there are no gaps between numbers. There is no answer to the question, "What is the next number bigger than two?" If a number, β, is bigger than two, then there is a number bigger than two and less than β. There is no number, c, bigger than two with no numbers between two and c. This remark is also true for fractions but not for integers. Three is bigger than two and there is no integer between two and three.

Numbers are rather neat and I can see why people are interested in them.

Further Consideration of Chapter 1

The immediate goal is to study objects in motion and in particular, falling objects. I think that the most direct way to do this is to watch objects fall and I council those who are interested in motion to take every opportunity to do so.

I can see motion take place but when I start to think about what is actually going on, moment to moment, point to point, I run into a wall. The ancient Greeks had much the same problem and their argument went something like this: A rock is thrown at a tree. The flight of the rock takes place during an interval of time which is composed of indivisible little chunks of time. The length of the path is the sum of the distances the rock travels during the chunks of time. These chunks of time can't have positive length because if they did, they could be divided in half and they are supposed to be indivisible. So the chunks of time have zero length and since the rock can't travel a positive distance in zero time, it must travel zero distance in a chunk of time. This means that the sum of these distances, the length of the path, is zero. It was hard to see how the visually evident fact of motion could happen.

But what is so wrong with indivisible chunks of time that have positive length? It seems to me that this is the way the Real World is in a practical sense. I can't deny, however, that I do find myself thinking that **if** I had a better watch, I could cut that next little piece of time in two. Could it be possible that there are positive instants of time that can't be cut in two? Would there be a problem, practical or otherwise, in supposing that the indivisible chunks of time are 1/10 second long, or a second long, or a minute long, or 10,000 years long?. What about if they were 10^{-50} seconds long?

Time fascinates me and I think about it a lot. I find subjective time particularly interesting.
A man who was putting up my television antenna told me that he had been a torpedo bomber pilot in World War II flying off an aircraft carrier. He said that his plane had been damaged on a mission and that he had crashed when he landed on his return to the carrier. His plane ended up teetering on the edge of the flight deck with the nose out over the ocean. His radio man got out and then he remembers getting very carefully out of the cockpit, very carefully inching his way back over the wing, and very carefully getting down onto the deck. His radio man walked up to him and said, "Man, I never saw anyone get out of a plane faster than that." Two people had two very different subjective times for the same event.

The person late for an appointment and the person waiting view the same length of time quite differently. One man's fleeting moment is another man's eternity.

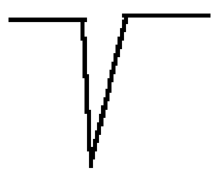
When I was five years old, the summer seemed to last forever but at fifty, the summer was gone quicker than you could say "knife." My explanation is that at the age of five a

summer was one twentieth of my life while at the age of fifty, it is a two-hundredth of my life and my life is what I have to measure time against.
Of course I always have objective time. I look at my watch and discover that I have only been waiting five minutes, not twenty.

Speaking of clocks, I wonder if it would be possible to make an alarm clock that would ring at midnight on January 1, one million years from January 1, 2000.

It seems to me that my view of the world at large is subjective. I tend to relate everything to me. I have lived most of my life in the western United States and I think of the texture of the earth as being pretty rough. I look up at the Rocky Mountains and think that they are a pretty big bump.

When I was a kid in the mountains of Idaho, I heard of bottomless lakes. I realized that they weren't really bottomless but the picture I had in mind was a profile where the depth of the lake was greater than the length.



In later years I reflected that this picture was probably not correct. 'Bottomless' probably meant that no fisherman had gone out on the lake with enough line to reach the bottom. The lake was about a mile above sealevel and I doubted that the bottom extended down that far. It was about 20 miles long and if I used a mile as the upper bound of its depth, the picture would look more like

I began thinking what a scale model of Earth would look like if its diameter was 6 inches. The circumference of the earth is about 24,000 miles, the highest mountains are about 6 miles high and the deepest trenches in the ocean are about 6 miles deep.

How big would the Himalayan Mountains be on the model? How deep would the Marianas trench be? How deep would the ocean be on this model? How thick would the tectonic plates be? How big a bump would the Rocky Mountains make? How big of a flash of light would a hydrogen bomb make? How high off of the model Earth would a space shuttle fly? As I began to answer these questions, I realized that I thought the earth was wrinkled because I was so small. The solar system thinks the earth is a smooth as a billiard ball.

I like the idea of scale models and I thought of making a scale model of the solar system in my back yard. It was impossible to scale. If my sun was big enough to be seen, Pluto was out of sight. It may not be entirely trivial for the invading aliens to find our planet.

The ‘Tower’ experiment, which is often attributed to Galileo, was apparently performed by Simon Stevin. He dropped spheres of unequal weight from the Tower of Pisa, which is about 180 feet high and it was said that they fell within three or four finger's breadth of each other. Even though they didn’t hit exactly together it was evidently close enough to convince Galileo that under ideal circumstances they would. ‘Close’ seems to count in horse shoes, hand grenades, and Real World experimentation.

If Galileo’s conjecture is correct and the balls didn’t hit the ground together, then the balls must have been dropped from different heights or at different times. I doubt that the person dropping the balls would make a three inch mistake at the top so they must have been dropped at different times. I wonder how different the times would have to be to give a three inch difference in distance after 180 feet? After I have found a relation between
the time an object falls and the distance it falls, I can answer this question.

Newton claims that between two objects there is an attractive force that depends on a property of the objects called <u>gravitational mass</u>. The force of attraction is proportional to the product of their gravitational masses and inversely proportional to the square of the distance between them.

Newton has also patiently explained to me that if I want to change the motion of a body, I must apply force. The amount of force necessary to change the motion depends on a property of the object called <u>inertial mass</u>. I can recall trying to stop my car from rolling backward down a slight incline by standing behind it and pushing. My pushing was futile in the face of the terrible inexoribility of inertial mass.

A consequence of Galileo's experiment is that the inertial mass of an object equals the gravitational mass of the object. I think that is amazing. Why should the property of matter that resists a change in its motion be numerically equal to the property of matter that attracts other matter? I might mention that these are both equal to the property of matter that appears as ‘m’ in the expression, $e=mc^2$.

Kepler analyzed data and gave a model of the solar system which verified Newton’s theory. Michelson and Morley used better measuring instruments and came up with data which led Einstein to a new model of physics. The Einstein model was necessary for high speeds and it reduced to the Newton model if the speed was moderate. Each model had a place where it could shine. This is an example of a model growing harmoniously in the light of new information.

Sometimes models are replaced by other, quite different models. The model that explained ‘thunder’ as the upsetting of God's apple cart, has largely been replaced by clouds bumping together.

It is not uncommon for a new model to contend with the models that preceded it. Evolution did not replace the Adam and Eve model, it contends with it. Some models are extremely resistant to change.

The decision as to whether something is in the Ideal World or the Real World is not often clear and metaphysics is a popular ground for border disputes. I also wonder if political, economic, and social theories are in the Real World or the Ideal World. Are communism and capitalism systems for the Ideal World or the Real World?

Infinity is a concept that I put firmly in the Ideal World and I always find it fun to think about this strange idea. It seems to bring the metaphysical eternity onto the playing field. This is one of my favorite examples.

I have an infinite stack of one dollar bills. Their serial numbers start with 1 and are numbered in order using all the positive whole numbers. I play a card game in which I lose $100 every hand and I play forever. When I lose the first hand, I pay my $100 but I put dollar bill #1 in my pocket for sentimental reasons. When I lose the second hand, I pay my $100 but put dollar bill #2 in my pocket for sentimental reasons. I may have to pay $101 and get my #2 back in change but I am willing to go to the trouble and my opponent is agreeable. I continue playing in this way. At the end of the nth hand, I put dollar bill #n in my pocket for sentimental reasons.

And at the end of this infinite night of losing poker, I walk away with all the money. This is clearly not Real World.